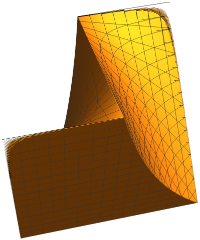
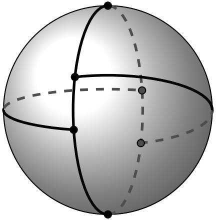
Topology of totally positive spaces

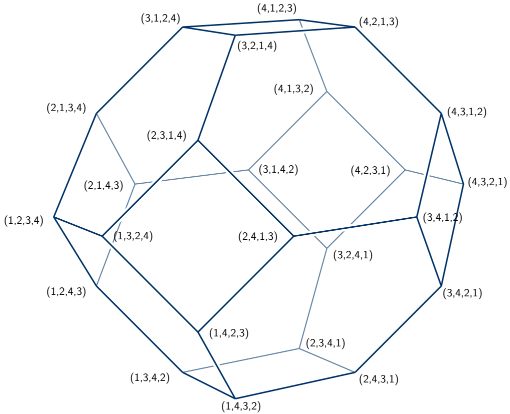
Slides available at `lacim.uqam.ca/~snkarp`

Steven N. Karp, LaCIM, Université du Québec à Montréal
joint work with Pavel Galashin and Thomas Lam
arXiv:1707.02010, 1801.08953, 1904.00527

ALGECOM-XIX
University of Illinois at Urbana–Champaign

Permutohedron

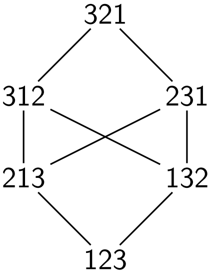


- The vertices of the permutohedron are $(\pi(1), \cdots, \pi(n)) \in \mathbb{R}^n$ for $\pi \in \mathfrak{S}_n$.
- The edges of the permutohedron are

$$(\cdots, i, \cdots, i+1, \cdots) \quad \longleftrightarrow \quad (\cdots, i+1, \cdots, i, \cdots).$$

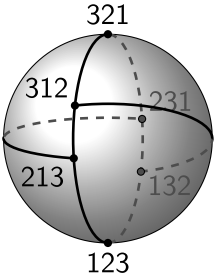
These correspond to cover relations in the *weak Bruhat order* on $\mathfrak{S}_n$.

Permutohedron for the strong Bruhat order?



$\mathfrak{S}_3$ (strong order)

$\rightsquigarrow$



?

- Using *total positivity*, we can define a space whose d-dimensional faces correspond to intervals of length d in the strong Bruhat order on $\mathfrak{S}_n$.
- This space is not a polytope! However, topologically it is just as good:
  1. it is partitioned into faces F, each homeomorphic to an open ball;
  2. the boundary ∂F of each face F is a union of lower-dimensional faces;
  3. the closure $\overline{F}$ of each face F is homeomorphic to a closed ball[1].

Such a space is called a *regular CW complex*.

[1] via a homeomorphism which sends F to the interior of the closed ball

Introduction to total positivity

- A matrix is *totally positive* if every submatrix has positive determinant.

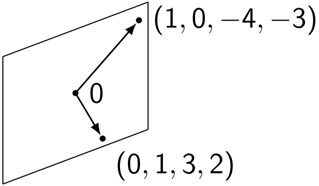
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 71.5987\cdots \\ \lambda_2 &= 3.6199\cdots \\ \lambda_3 &= 0.7168\cdots \\ \lambda_4 &= 0.0646\cdots \end{aligned}$$

- Gantmakher, Krein (1937): the eigenvalues of a square totally positive matrix are all real, positive, and distinct.
- Totally positive matrices are a discrete analogue of *totally positive kernels* (e.g. $K(x,y) = e^{xy}$), introduced by Kellogg (1918).
- Lusztig (1994): total positivity for algebraic groups G (e.g. $G = \mathrm{SL}_n$) and partial flag varieties G/P (e.g. $G/P = \mathrm{Gr}_{k,n}$, Fl_n).
- Fomin, Zelevinsky (2002): cluster algebras.
- Postnikov (2006): *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$. It has been related to the ASEP, the KP equation, Poisson geometry, quantum matrices, scattering amplitudes, mirror symmetry, singularities of curves, ...

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

$V :=$



$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

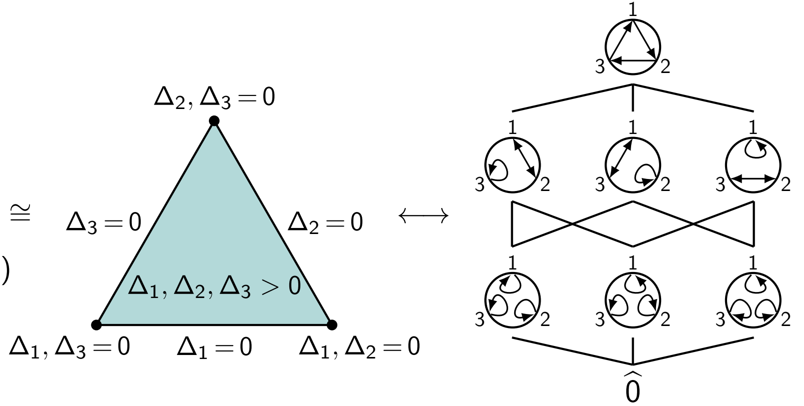
$$\Delta_{12} = 1, \quad \Delta_{13} = 3, \quad \Delta_{14} = 2, \quad \Delta_{23} = 4, \quad \Delta_{24} = 3, \quad \Delta_{34} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \cdots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call $V \in \mathrm{Gr}_{k,n}$ *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all such V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.
- $\mathrm{Gr}_{1,n}$ is projective space $\mathbb{P}^{n-1}$, and its totally nonnegative part is a simplex. We can think of $\mathrm{Gr}_{k,n}^{\geq 0}$ as the Grassmannian notion of a simplex.

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a decomposition into cells (open balls) due to Rietsch (1998) and Postnikov (2006). Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$ $(= \mathbb{P}^2_{\geq 0})$

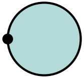
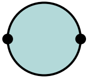


- Postnikov showed that the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is given by *circular Bruhat order* on *decorated permutations* with k *anti-excedances*.

The topology of $\mathrm{Gr}_{k,n}^{\geq 0}$

Conjecture (Postnikov (2006))

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ is a regular CW complex. Thus the closure of every cell is homeomorphic to a closed ball.

- e.g.  non-regular CW complex  regular CW complex
- Williams (2007): The face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is graded, thin, and shellable.
- Postnikov, Speyer, Williams (2009): $\mathrm{Gr}_{k,n}^{\geq 0}$ is a CW complex.
- Rietsch, Williams (2010): Postnikov's conjecture is true up to homotopy.
- Galashin, Karp, Lam (2017): $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a closed ball.

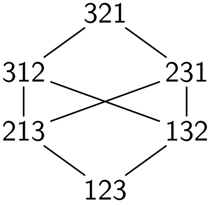
Theorem (Galashin, Karp, Lam)

Postnikov's conjecture is true.

- We prove more generally that the cell decomposition of $(G/P)_{\geq 0}$ is a regular CW complex, confirming a conjecture of Williams (2007).

Motivation 1: combinatorics of regular CW complexes

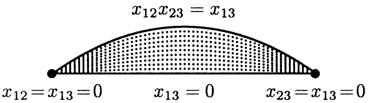
- Any convex polytope (decomposed into faces) is a regular CW complex.
- Björner (1984): Every regular CW complex is uniquely determined by its face poset (up to homeomorphism). Conversely, any poset which is *graded*, *thin*, and *shellable* is the face poset of some regular CW complex.

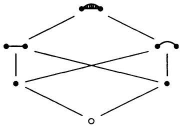


$\mathfrak{S}_3$ (strong order)

$$\mathrm{link}_{I_3}(U_3^{\geq 0}) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : \begin{matrix} a + c = 1, \\ \text{all minors} \geq 0 \end{matrix} \right\}$$

$\leftsquigarrow$

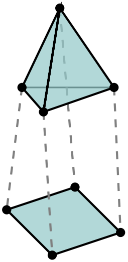




- Edelman (1981): $\mathfrak{S}_n$ is graded, thin, and shellable.
- Björner (1984): Is there a 'natural' regular CW complex with face poset $\mathfrak{S}_n$?
- Fomin and Shapiro (2000) conjectured that $\mathrm{link}_{I_n}(U_n^{\geq 0})$ is such a regular CW complex. This was proved by Hersh (2014), in general Lie type. We give a new proof of Hersh's theorem.

Motivation 2: amplituhedra and Grassmann polytopes

- By definition, a polytope is the image of a simplex under an affine map:



simplex in $\mathbb{P}^{n-1}$ $\dashrightarrow$ $\mathrm{Gr}_{k,n}^{\geq 0}$

linear map $\mathbb{R}^n \to \mathbb{R}^{m+1}$ $\qquad$ linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$

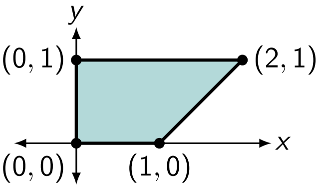
polytope in $\mathbb{P}^m$ $\dashrightarrow$ *Grassmann polytope* in $\mathrm{Gr}_{k,k+m}$

A *Grassmann polytope* is the image of a map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

- When the matrix Z has positive maximal minors, the Grassmann polytope is called an *amplituhedron*. Amplituhedra generalize cyclic polytopes ($k = 1$) and totally nonnegative Grassmannians ($k + m = n$). They were introduced by the physicists Arkani-Hamed and Trnka (2014), and inspired Lam (2015) to define Grassmann polytopes.

Motivation 2: amplituhedra and Grassmann polytopes

- Arkani-Hamed, Bai, Lam (2017): a *positive geometry* is a space equipped with a *canonical differential form*, which has logarithmic singularities at the boundaries of the space. Examples include convex polytopes:



$$\pm \frac{(1+y)dxdy}{xy(1-y)(1-x+y)}$$

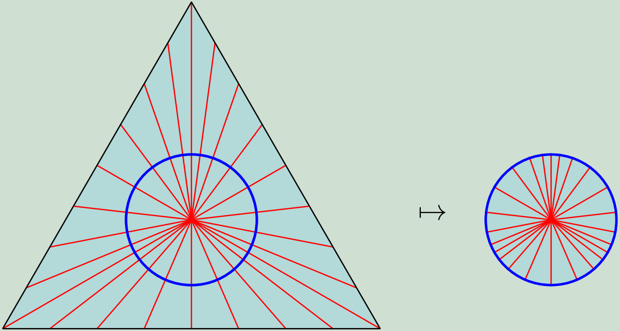
- The amplituhedron is conjecturally a positive geometry, whose canonical form for $m = 4$ is the tree-level scattering amplitude in planar $\mathcal{N} = 4$ SYM.
- Intuition from physics: the geometry determines the canonical form, and vice-versa. In order to understand amplituhedra (and more generally, Grassmann polytopes), we first need to understand $\mathrm{Gr}_{k,n}^{\geq 0}$.
- Other physically relevant positive geometries include *associahedra*, *cosmological polytopes*, *halohedra*, *accordiohedra*, ...

Technique 1: contractive flows

Theorem

Every compact, convex subset of $\mathbb{R}^d$ is homeomorphic to a closed ball.

Proof



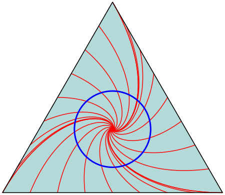
- This proof does not directly work for $\mathrm{Gr}_{k,n}^{\geq 0}$, since it is not *totally geodesic*.

Cyclic symmetry of $\mathrm{Gr}_{k,n}^{\geq 0}$

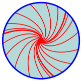
- The space $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cyclic symmetry, coming from the *cyclic action*

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \mapsto \begin{bmatrix} | & & | & | \\ v_2 & \cdots & v_n & (-1)^{k-1} v_1 \\ | & & | & | \end{bmatrix}.$$

- This action gives a vector field on $\mathrm{Gr}_{k,n}^{\geq 0}$ with a global attractor. The integral curves yield a homeomorphism from $\mathrm{Gr}_{k,n}^{\geq 0}$ to a closed ball, as above.
- e.g. $\mathrm{Gr}_{1,3}^{\geq 0}$



$\mapsto$



- A similar argument shows the following spaces are homeomorphic to closed balls: *cyclically symmetric* amplituhedra, Lam's compactified space of electrical networks, Lusztig's $(G/P)_{\geq 0}$, and Huang and Wen's totally nonnegative orthogonal Grassmannian.

The complete flag variety Fl_n

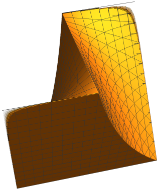
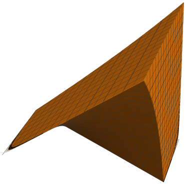
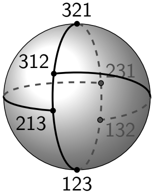
- Another instance of G/P is the *complete flag variety* Fl_n, the set of tuples

$$\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{R}^n, \qquad \text{where } V_k \in \mathrm{Gr}_{k,n} \text{ for all } k.$$

- Lusztig (1994): $\mathrm{Fl}_n^{\geq 0}$ is the subset where $V_k \in \mathrm{Gr}_{k,n}^{\geq 0}$ for all k.
- e.g. $\mathrm{Fl}_3^{\geq 0}$ consists of complete flags $\{0\} \subset V_1 \subset V_2 \subset \mathbb{R}^3$ such that V_1 is spanned by a vector (a_1, a_2, a_3), and V_2 is orthogonal to $(b_1, -b_2, b_3)$, with

$$a_1 b_1 - a_2 b_2 + a_3 b_3 = 0, \qquad a_1, a_2, a_3, b_1, b_2, b_3 \geq 0.$$

This space has 4 facets, given by setting one of a_1, b_1, a_3, b_3 to 0.

- Lusztig (1994), Rietsch (1999): $\mathrm{Fl}_n^{\geq 0}$ has a cell decomposition whose d-dimensional cells are indexed by intervals of length d in $(\mathfrak{S}_n, \leq_{\text{strong}})$.

Technique 2: links and the Fomin–Shapiro atlas

- Unfortunately, $\mathrm{Gr}_{k,n}^{\geq 0}$ has cells which do not admit any contractive flow. Therefore we need a new technique to show it is a regular CW complex.
- Brown (1962), Smale (1961), Freedman (1982), Perelman (2003):

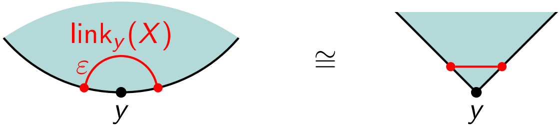
Theorem (consequence of generalized Poincaré conjecture)

Suppose that X is a compact topological manifold with boundary, whose interior X° is contractible and whose boundary ∂X is homeomorphic to a sphere. Then X is homeomorphic to a closed ball.

- We want to apply this result when X is the closure of a cell of $\mathrm{Gr}_{k,n}^{\geq 0}$.
- Rietsch (1999), Postnikov (2006): X° is homeomorphic to an open ball.
- Williams (2007): The face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is graded, thin, and shellable. By induction, ∂X is a regular CW complex. Therefore by results of Björner (1984), ∂X is homeomorphic to a sphere.
- Note: the conclusion of the theorem follows from just the result of Brown and the generalized Schoenflies theorem of Mazur (1959) and Brown (1960), if we also know that X° is homeomorphic to an open ball.

Link induction

- We want to show that X is a topological manifold with boundary, i.e. X looks like a closed half-space in $\mathbb{R}^d$ near any point on its boundary.
- We use the framework of *links* introduced by Fomin and Shapiro (2000).



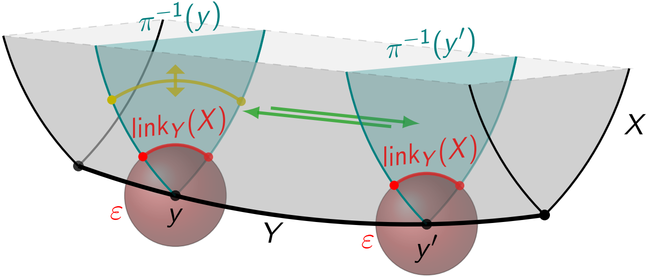
- It suffices to prove that:
  1. $\text{link}_y(X)$ is homeomorphic to a closed ball; and
  2. locally near y, the space X looks like the cone over $\text{link}_y(X)$.
- We prove (1) by a similar induction. In order that this not reduce to *another* induction, we must be more subtle: we define $\text{link}_Y(X)$, a space of dimension $\dim(X) - \dim(Y) - 1$, where $Y \subset X$ is the cell containing y.
- We have

$$\text{link}_Y(X) = \bigsqcup_{Y \subset Z \subseteq X} \text{link}_Y^\circ(Z),$$

and the link of $\text{link}_Y(Z)$ inside $\text{link}_Y(X)$ is homeomorphic to $\text{link}_Z(X)$.

Fomin–Shapiro atlas

- To define $\mathrm{link}_Y(X)$, we need a projection $\pi : X \to Y$ and translations $\pi^{-1}(y') \to \pi^{-1}(y)$. To get (2), we need dilation actions on $\pi^{-1}(y)$.



- Fomin and Shapiro constructed the projections and translations for $U_n^{\geq 0}$ via matrix factorizations, using work of Kazhdan and Lusztig (1980).
- e.g. $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & \frac{ac-b}{a} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \pi\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & \frac{ac-b}{a} \\ 0 & 0 & 1 \end{bmatrix}.$

1. How do we define the dilation actions?
2. How do we construct these maps for $\mathrm{Gr}_{k,n}^{\geq 0}$?

} *Snider's embedding* (2011)

Snider's embedding

- We fix I, and embed the subset of $\mathrm{Gr}_{k,n}$ where $\Delta_I \neq 0$ into the *affine flag variety* $\widetilde{\mathrm{Fl}}_n$, the set of n-periodic matrices modulo certain row operations.
- e.g. Let $I = \{1,3\}$ with $k=2, n=4$. Then Snider's embedding is

$$\begin{bmatrix} 1 & a & 0 & b \\ 0 & c & 1 & d \end{bmatrix} \mapsto \begin{matrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & \\ \cdots & 0 & a & 0 & b & 1 & 0 & \cdots & & & & & \\ & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & \\ & & \cdots & 0 & d & 0 & c & 1 & 0 & \cdots & & & \\ & & & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & & \\ & & & & \cdots & 0 & a & 0 & b & 1 & 0 & \cdots & \\ & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{matrix} .$$

- We define the projection and translation maps by 'matrix factorizations' in $\widetilde{\mathrm{Fl}}_n$, which were studied by Knutson, Woo, and Yong (2013) for Fl_n.
- We obtain the dilation actions by translating to a 'hidden' point in $\widetilde{\mathrm{Fl}}_n$ in the closure of the image of Y (the analogue of a permutation matrix).
- For arbitrary G/P, we construct a generalization of Snider's embedding. A similar embedding was independently found by Huang (2019).

Open problems

- Show that the following spaces are regular CW complexes:
  1. amplituhedra;
  2. Fomin and Zelevinsky's double Bruhat cells;
  3. Lam's compactified space of electrical networks;
  4. Galashin and Pylyavskyy's cell decomposition of the totally nonnegative orthogonal Grassmannian;
  5. Rietsch's totally nonnegative part of a Peterson variety;
  6. He's cell decomposition of pieces of the wonderful compactification.
- Show that the interior of a link arising in $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to an open ball (and so avoid the use of the generalized Poincaré conjecture).
- Study total positivity in Kac–Moody groups and flag varieties.
- Study the topology of Grassmann polytopes.

Thank you!