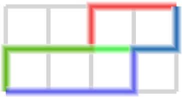
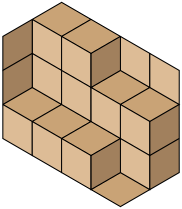
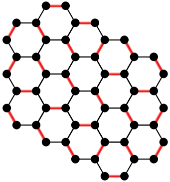
Geometry of the amplituhedron

Slides available at `www-personal.umich.edu/~snkarp`

 $\leftrightarrow$  $\leftrightarrow$ 

Steven N. Karp, University of Michigan

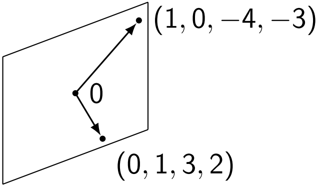
`arXiv:1608.08288` (joint with Lauren Williams)
`arXiv:1708.09525` (joint with Lauren Williams and Yan Zhang)

October 21st, 2018
AMS special session on cluster algebra, Poisson geometry, and related topics, University of Michigan

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces V of $\mathbb{R}^n$.

$V :=$



$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}$$

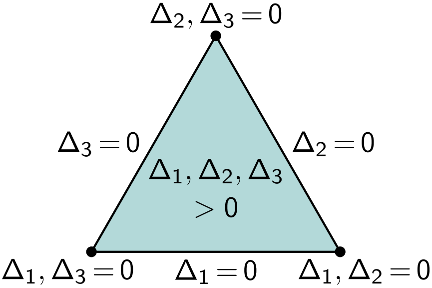
$$\Delta_{12}(V) = 1 \qquad \Delta_{13}(V) = 3 \qquad \Delta_{14}(V) = 2$$
$$\Delta_{23}(V) = 4 \qquad \Delta_{24}(V) = 3 \qquad \Delta_{34}(V) = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \dots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call V *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all such V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.

The positroid cell decomposition $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition due to Rietsch (1999) and Postnikov (2007). Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

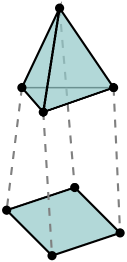
$\mathrm{Gr}_{1,3}^{\geq 0} = \mathbb{P}_{\geq 0}^{2} \cong$



- $\mathrm{Gr}_{1,n}^{\geq 0}$ is an $(n-1)$-dimensional simplex in $\mathbb{P}^{n-1}$. So, one can think of $\mathrm{Gr}_{k,n}^{\geq 0}$ as a generalization of a simplex into the Grassmannian.
- Each cell can be parametrized using a *plabic graph*, whose dual quiver conjecturally gives a cluster structure on the associated *positroid variety*.

Amplituhedra and Grassmann polytopes

- By definition, a polytope is the image of a simplex under an affine map:



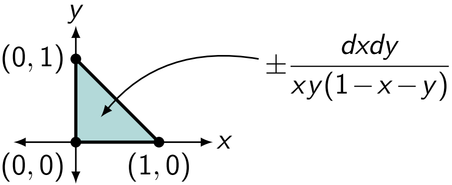
$$\begin{tikzcd}
\text{simplex in } \mathbb{P}^{n-1} \arrow[r, dashed] \arrow[d, mapsto, "\substack{\text{linear map} \\ \mathbb{R}^n \to \mathbb{R}^{m+1}}"] & \mathrm{Gr}_{k,n}^{\geq 0} \arrow[d, mapsto, "\substack{\text{linear map} \\ \mathbb{R}^n \to \mathbb{R}^{k+m}}"] \\
\text{polytope in } \mathbb{P}^{m} \arrow[r, dashed] & \substack{\textit{Grassmann polytope} \\ \text{in } \mathrm{Gr}_{k,k+m}}
\end{tikzcd}$$

A *Grassmann polytope* is the image of a map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

- When the matrix Z has positive maximal minors, the Grassmann polytope is called an *amplituhedron*, denoted $\mathcal{A}_{n,k,m}(Z)$. Amplituhedra generalize cyclic polytopes ($k = 1$) and totally nonnegative Grassmannians ($k + m = n$). They were introduced by Arkani-Hamed and Trnka (2014), and inspired Lam (2015) to define Grassmann polytopes.

Positive geometries and canonical forms

- Arkani-Hamed, Bai, Lam (2017): a *positive geometry* is a space equipped with a *canonical differential form*, which has logarithmic singularities at the boundaries of the space. Examples include convex polytopes:



- The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is conjecturally a positive geometry, whose canonical form for $m = 4$ is the tree-level scattering amplitude in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory.
- Other positive geometries with physically relevant canonical forms include *associahedra*, *Cayley polytopes*, and *cosmological polytopes*.
- Arkani-Hamed, Bai, He, Yan and Bazier-Matte, Douville, Mousavand, Thomas, Yıldırım (2018): new construction of generalized associahedra and Newton polytopes of F-polynomials in finite simply-laced types.

Canonical form of a cluster variety

- $\mathrm{Gr}_{k,n}^{\geq 0}$ is a positive geometry. The canonical form of $\mathrm{Gr}_{2,4}^{\geq 0}$ is (up to sign)

$$\frac{d\Delta_{23}}{\Delta_{23}}\frac{d\Delta_{34}}{\Delta_{34}}\frac{d\Delta_{14}}{\Delta_{14}}\frac{d\Delta_{13}}{\Delta_{13}}, \text{ where } \{\Delta_{12}=1, \Delta_{23}, \Delta_{34}, \Delta_{14}, \Delta_{13}\} \text{ is a cluster.}$$

- The form does not depend (up to sign) on the choice of cluster.

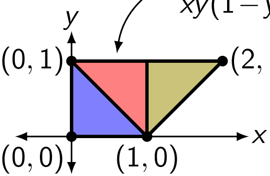
$$\begin{aligned} xx' &= P + Q \\ xdx' + x'dx &= dP + dQ \\ \frac{dx'}{x'} &= -\frac{dx}{x} + \frac{dP}{xx'} + \frac{dQ}{xx'} \qquad (dy \wedge dy = 0) \end{aligned}$$

Conjecture (Arkani-Hamed, Bai, Lam (2017))

Let $X = \mathrm{Spec}(A)$ be the cluster variety of a 'reasonable' cluster algebra A. Then we can compactify X so that its nonnegative part is a positive geometry with canonical form $\pm \bigwedge_{i=1}^{n} \frac{dx_i}{x_i}$ for any cluster $\{x_1, \dots, x_n\}$ of A.

Triangulations

- One way to find the canonical form of a positive geometry is by triangulating it into simpler pieces:



$$\frac{(1+y)dxdy}{xy(1-y)(1-x+y)} = \frac{dxdy}{xy(1-x-y)} + \frac{dxdy}{(1-x)(1-y)(x+y-1)} + \frac{dxdy}{(x-1)(1-y)(1-x+y)}$$

Conjecture (Arkani-Hamed, Trnka (2014))

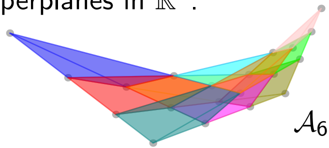
The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is triangulated by the images of certain $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, coming from the BCFW recursion.

Problem

Find a 'triangulation-independent' description of the amplituhedron form.

Triangulating amplituhedra: progress so far

- Sturmfels (1988): When $k = 1$, any amplituhedron $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope* with n vertices in $\mathbb{P}^m$, i.e. it is combinatorially equivalent to a polytope whose vertices lie on the *rational normal curve* $(1 : t : \cdots : t^m)$. Triangulations of cyclic polytopes are already interesting!
- Karp, Williams (2018): When $m = 1$, any amplituhedron $\mathcal{A}_{n,k,1}(Z)$ is isomorphic to the complex of bounded faces of a *cyclic hyperplane arrangement* of n hyperplanes in $\mathbb{R}^k$.



$\mathcal{A}_{6,3,1}(Z)$

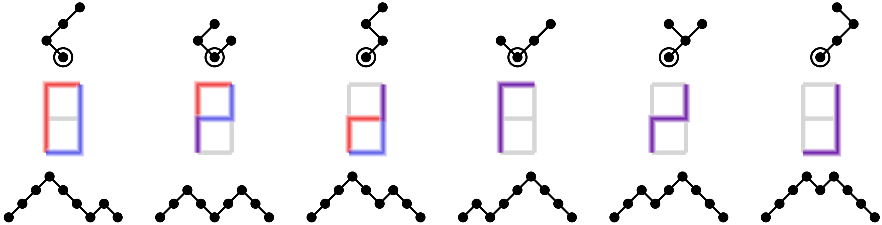
- Karp, Williams, Zhang (2018): the pieces of the conjectured triangulation of $\mathcal{A}_{n,k,4}(Z)$ are disjoint when $k = 2$.
- The proofs of these results involve a careful study of sign vectors.
- We know that $\mathcal{A}_{n,k,m}(Z)$ is homeomorphic to a closed ball when $k = 1$, $m = 1$, or $n - k - m = 1$, and in certain special cases. In general, it is not even known whether $\mathcal{A}_{n,k,m}(Z)$ is contractible.

Combinatorics of triangulations

Conjecture (Arkani-Hamed, Trnka (2014))

The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is triangulated by the images of certain $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, coming from the BCFW recursion.

- The number of top-dimensional cells in a BCFW triangulation is the *Narayana number* $N_{n-3,k+1} = \frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$.
- e.g. For $n = 7$, $k = 2$, we have $N_{7-3,2+1} = 6$:



- $k = 1$, m even: every triangulation of $\mathcal{A}_{n,1,m}(Z)$ has $\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$ top cells.
- $m = 2$: there is a nice triangulation of $\mathcal{A}_{n,k,2}(Z)$ with $\binom{n-2}{k}$ top cells.

Combinatorics of triangulations: plane partitions?

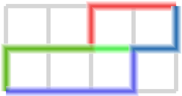
- Define the *MacMahon number*

$$M(a,b,c) := \prod_{p=1}^{a} \prod_{q=1}^{b} \prod_{r=1}^{c} \frac{p+q+r-1}{p+q+r-2}.$$

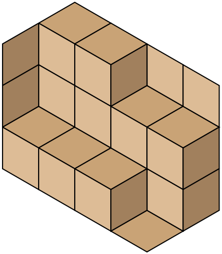
Conjecture (Karp, Williams, Zhang (2018))

For m even, there exists a cell decomposition of $\mathcal{A}_{n,k,m}(Z)$ with $M(k, n-k-m, \frac{m}{2})$ top-dimensional cells.

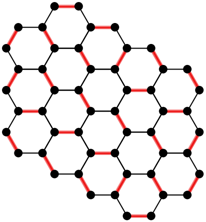
- $M(a,b,c)$ is the number of *plane partitions* inside an $a \times b \times c$ box.
- e.g. $M(2,4,3) = 490$:



$\leftrightarrow$



$\leftrightarrow$



Combinatorics of triangulations: plane partitions?

- Define the *MacMahon number*

$$M(a,b,c) := \prod_{p=1}^{a}\prod_{q=1}^{b}\prod_{r=1}^{c}\frac{p+q+r-1}{p+q+r-2}.$$

Conjecture (Karp, Williams, Zhang (2018))

For m even, there exists a cell decomposition of $\mathcal{A}_{n,k,m}(Z)$ with $M(k, n-k-m, \frac{m}{2})$ top-dimensional cells.

Problem

Interpret properties of plane partitions in terms of amplituhedra.

- The $k \leftrightarrow n-k-m$ symmetry comes (for $m=4$) from *parity* of the scattering amplitude. Galashin and Lam (2018) showed that the *stacked twist map* interchanges triangulations of $\mathcal{A}_{n,k,m}(Z)$ and $\mathcal{A}_{n,n-k-m,m}(Z')$.

Problem

Explain the conjectural symmetry for amplituhedra between k and $\frac{m}{2}$.