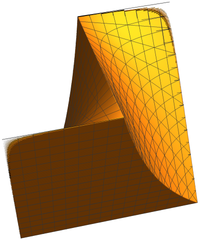
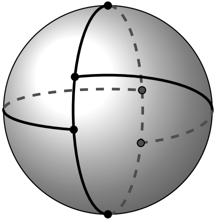
Regularity theorem for totally nonnegative flag varieties

Slides available at `lacim.uqam.ca/~snkarp`





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AMS special session on cluster algebras and plabic graphs

Total positivity

- A matrix is *totally positive* if every submatrix has positive determinant.

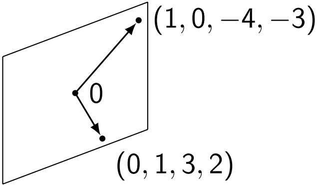
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 71.5987\cdots \\ \lambda_2 &= 3.6199\cdots \\ \lambda_3 &= 0.7168\cdots \\ \lambda_4 &= 0.0646\cdots \end{aligned}$$

- Gantmakher, Krein (1937): the eigenvalues of a square totally positive matrix are all real, positive, and distinct.
- Totally positive matrices are a discrete analogue of *totally positive kernels* (e.g. $K(x,y) = e^{xy}$), introduced by Kellogg (1918).
- Lusztig (1994): total positivity for algebraic groups G (e.g. $G = \mathrm{SL}_n$) and partial flag varieties G/P (e.g. $G/P = \mathrm{Gr}_{k,n}$).
- Fomin, Zelevinsky (2002): cluster algebras.
- Postnikov (2006): *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$. It has been related to the ASEP, the KP equation, Poisson geometry, quantum matrices, scattering amplitudes, mirror symmetry, singularities of curves, ...

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

$V :=$



$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

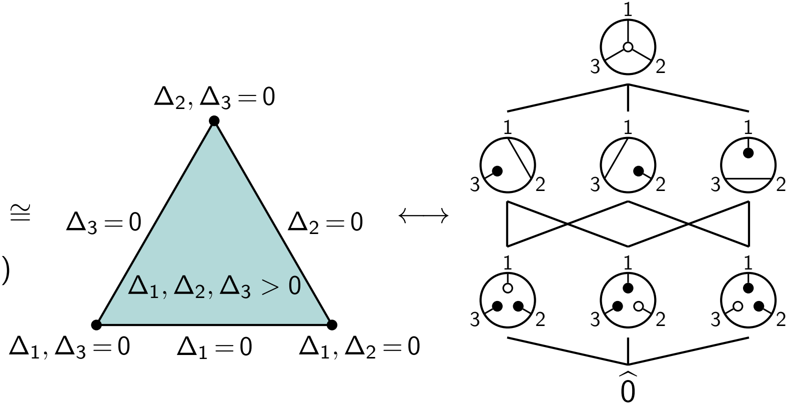
$$\Delta_{12} = 1, \;\; \Delta_{13} = 3, \;\; \Delta_{14} = 2, \;\; \Delta_{23} = 4, \;\; \Delta_{24} = 3, \;\; \Delta_{34} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \cdots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call $V \in \mathrm{Gr}_{k,n}$ *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all such V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.
- $\mathrm{Gr}_{1,n}$ is projective space $\mathbb{P}^{n-1}$, and its totally nonnegative part is a simplex. We can think of $\mathrm{Gr}_{k,n}^{\geq 0}$ as the Grassmannian notion of a simplex.

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a decomposition into cells (open balls) due to Rietsch (1998) and Postnikov (2006). Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$ $(= \mathbb{P}^2_{\geq 0})$



- Postnikov showed that the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is given by *plabic graphs* (up to move equivalence) under removal of edges.

The topology of $\mathrm{Gr}_{k,n}^{\geq 0}$

Conjecture (Postnikov (2006))

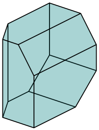
The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ is a regular CW complex. Thus the closure of every cell is homeomorphic to a closed ball.

- A *regular CW complex* is a space satisfying the following properties:
  1. it is partitioned into cells F, each homeomorphic to an open ball;
  2. the boundary ∂F of each cell F is a union of lower-dimensional cells;
  3. the closure $\overline{F}$ of each cell F is homeomorphic to a closed ball[1].

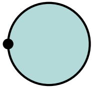
[1] via a homeomorphism which sends F to the interior of the closed ball

A regular CW complex is a natural generalization of a convex polytope.

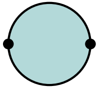
- e.g.



regular
CW complex



non-regular
CW complex



regular
CW complex

The topology of $\text{Gr}_{k,n}^{\geq 0}$

Conjecture (Postnikov (2006))

The cell decomposition of $\text{Gr}_{k,n}^{\geq 0}$ *is a regular CW complex. Thus the closure of every cell is homeomorphic to a closed ball.*

- Lusztig (1998): $\text{Gr}_{k,n}^{\geq 0}$ is contractible.
- Williams (2007): The face poset of $\text{Gr}_{k,n}^{\geq 0}$ is graded, thin, and shellable.
- Postnikov, Speyer, Williams (2009): $\text{Gr}_{k,n}^{\geq 0}$ is a CW complex (via *matching polytopes* of plabic graphs).
- Rietsch, Williams (2010): Postnikov's conjecture is true up to homotopy (via discrete Morse theory).
- Galashin, Karp, Lam (2017): $\text{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a closed ball.

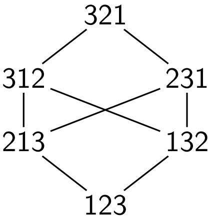
Theorem (Galashin, Karp, Lam)

Postnikov's conjecture is true.

- Our result holds for all G/P, confirming a conjecture of Williams (2007).

Motivation 1: combinatorics of regular CW complexes

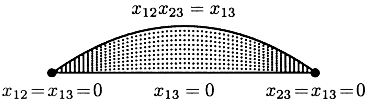
- Björner (1984): Every regular CW complex is uniquely determined by its face poset (up to homeomorphism). Conversely, any poset which is *graded*, *thin*, and *shellable* is the face poset of some regular CW complex.

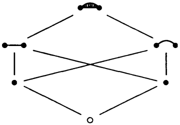


$\mathfrak{S}_3$ (Bruhat order)

$\leftrightsquigarrow$

$$\mathrm{link}_{I_3}(U_3^{\geq 0}) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : \begin{array}{l} a + c = 1, \\ \text{all minors} \geq 0 \end{array} \right\}$$

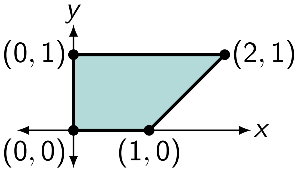




- Edelman (1981): $\mathfrak{S}_n$ is graded, thin, and shellable.
- Björner (1984): Is there a 'natural' regular CW complex with face poset $\mathfrak{S}_n$?
- Fomin and Shapiro (2000) conjectured that $\mathrm{link}_{I_n}(U_n^{\geq 0})$ is such a regular CW complex. This was proved by Hersh (2014), in general Lie type. We give a new proof of Hersh's theorem.

Motivation 2: scattering amplitudes and differential forms

- Arkani-Hamed, Bai, Lam (2017): we can associate to certain geometric spaces a *canonical differential form*.



$$\pm \frac{(1+y)dxdy}{xy(1-y)(1-x+y)}$$

$$\mathrm{Gr}_{k,n}^{\geq 0}$$

$$\pm \bigwedge_{x \in \text{cluster}} \frac{dx}{x}$$

- The differential form of the *amplituhedron*, a certain projection of $\mathrm{Gr}_{k,n}^{\geq 0}$, is conjecturally the tree-level scattering amplitude in planar $\mathcal{N} = 4$ SYM.
- Intuition from physics: the geometry determines the form, and vice-versa. In order to understand amplituhedra, first we need to understand $\mathrm{Gr}_{k,n}^{\geq 0}$.
- Other physically relevant spaces include *associahedra* and *accordiohedra*.

Open problems

- Show that the following spaces are regular CW complexes:
  1. Arkani-Hamed and Trnka's amplituhedra;
  2. Fomin and Zelevinsky's double Bruhat cells;
  3. totally nonnegative part of a Kac–Moody partial flag variety;
  4. Lam's compactified space of electrical networks;
  5. Galashin and Pylyavskyy's cell decomposition of the totally nonnegative orthogonal Grassmannian;
  6. Rietsch's totally nonnegative part of a Peterson variety;
  7. He's cell decomposition of the totally nonnegative part of the De Concini–Procesi wonderful compactification.

Thank you!