q-Whittaker functions, finite fields, and Jordan forms

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Combinatorial Algebra meets Algebraic Combinatorics

Schur functions

- A *partition* λ is a weakly-decreasing sequence of nonnegative integers.

- e.g. $\lambda = (4, 4, 1) =$

$T =$

1	3	3	4
4	4	8	8
5			

- A *semistandard tableau* T is a filling of λ with positive integers which is weakly increasing across rows and strictly increasing down columns.

Definition (Schur function)

$$s_\lambda(x_1, x_2, \ldots) := \sum_T \mathbf{x}^T,$$

where the sum is over all semistandard tableaux T of shape λ.

- $s_\lambda(\mathbf{x})$ is symmetric in the variables x_i.

Schur functions

- e.g. $s_{(2,1)}(x_1, x_2, x_3) =$

$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

- Schur functions appear in many contexts; for example, they:
  - form an *orthonormal basis* for the algebra of symmetric functions in $\mathbf{x}$;
  - are characters of the *irreducible polynomial representations* of $\mathrm{GL}_n(\mathbb{C})$;
  - give the values of the *irreducible characters* of the symmetric group S_n, when expanded in terms of power sum symmetric functions;
  - are representatives for *Schubert classes* in the cohomology ring of the Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{C})$;
  - define the *Schur processes* of Okounkov and Reshetikhin (2003).

Cauchy identity

Theorem (Cauchy)

$$\prod_{i,j\geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y})$$

- We can expand the left-hand side as a sum indexed by *nonnegative-integer matrices*, and the right-hand side as a sum indexed by pairs of *semistandard tableaux of the same shape*.
- e.g. Taking the coefficient of $x_1 x_2 y_1 y_2$ on each side gives

$$\begin{array}{ccccccc} 1 & + & 1 & = & 1 & + & 1 \\ 12 & & 21 & & \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right) & & \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \right) \end{array}$$

Burge correspondence (1974)

- The *Burge correspondence* (also known as *column Robinson–Schensted–Knuth*) is a bijection

$$M \mapsto (\mathrm{P}(M), \mathrm{Q}(M))$$

between nonnegative-integer matrices and pairs of semistandard tableaux of the same shape. It proves the Cauchy identity for Schur functions.

- $\mathrm{P}(M)$ is obtained via *column insertion* and $\mathrm{Q}(M)$ is obtained via *recording*.
- e.g. $w = 25143$

2	

2	
5	

1	2
5	

1	2
4	5

1	2	5
3	4	

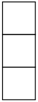
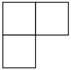
$\mathrm{P}(w)$

1	3	5
2	4	

$\mathrm{Q}(w)$

Nilpotent matrices

- An $n \times n$ matrix N over $\mathbb{k}$ is *nilpotent* if some power of N is zero. Such an N can be conjugated over $\mathbb{k}$ into *Jordan form*. Let $\mathrm{JF}^{\top}(N)$ be the *transpose* of the partition given by the sizes of the Jordan blocks.

- e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto$  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto$  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto$ 

- Algebraically, $\mathrm{JF}^{\top}(N)$ is the partition λ given by

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i = \dim(\ker(N^i)) \quad \text{for all } i.$$

Theorem (Gansner (1981))

Let N be a generic $n \times n$ strictly upper-triangular matrix, where $N_{i,j} = 0$ for all inversions (i,j) of w^{-1}. Then $\mathrm{P}(w)$ and $\mathrm{Q}(w)$ can be read off from the Jordan forms of the leading submatrices of N and $w^{-1}Nw$.

Burge correspondence via Jordan forms

- e.g. $w = 25143$ $\qquad N = \begin{bmatrix} 0 & {\color{red}0} & {\color{blue}a} & {\color{blue}b} & {\color{red}0} \\ 0 & 0 & {\color{blue}c} & {\color{blue}d} & {\color{blue}e} \\ 0 & 0 & 0 & {\color{red}0} & {\color{red}0} \\ 0 & 0 & 0 & 0 & {\color{red}0} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\qquad (a, b, c, d, e \in \Bbbk \text{ generic})$

P(w):

$$\begin{array}{c|c} & 1 \\ \hline 1 & 0 \end{array} \qquad \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 0 \\ 2 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & a \\ 2 & 0 & 0 & c \\ 3 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & a & b \\ 2 & 0 & 0 & c & d \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & a & b & 0 \\ 2 & 0 & 0 & c & d & e \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{|c|}\hline \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$$

Q(w):

$$\begin{array}{c|c} & 2 \\ \hline 2 & 0 \end{array} \qquad \begin{array}{c|cc} & 2 & 5 \\ \hline 2 & 0 & e \\ 5 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} & 2 & 5 & 1 \\ \hline 2 & 0 & e & 0 \\ 5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|cccc} & 2 & 5 & 1 & 4 \\ \hline 2 & 0 & e & 0 & d \\ 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & b \\ 4 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccccc} & 2 & 5 & 1 & 4 & 3 \\ \hline 2 & 0 & e & 0 & d & c \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & b & a \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{|c|}\hline \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$$

Flag variety

- A *complete flag* F in $\Bbbk^n$ is a sequence of nested subspaces

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = \Bbbk^n, \qquad \dim(F_i) = i \text{ for all } i.$$

- An $n \times n$ (nilpotent) matrix N is *strictly compatible* with F if

$$N(F_i) \subseteq F_{i-1} \quad \text{for all } i.$$

- The N in Gansner's theorem is precisely a matrix strictly compatible with two complete flags F and F' defined by

$$F_i := \langle \mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_i \rangle \quad \text{and} \quad F'_j := \langle \mathrm{e}_{w(1)}, \mathrm{e}_{w(2)}, \ldots, \mathrm{e}_{w(j)} \rangle.$$

The two sequences of matrices in the theorem are $(N|_{F_i})_{i=1}^n$ and $(N|_{F'_j})_{j=1}^n$.

- More generally, we can take any pair of flags (F, F') with *relative position* w, denoted $F \xrightarrow{w} F'$. The relative position records $\dim(F_i \cap F'_j)$ for all i and j, or alternatively, the *Schubert cell* of F' relative to F.

Burge correspondence via flags

Theorem (Steinberg (1976, 1988), Spaltenstein (1982), Rosso (2012))

Fix partial flags F and F' with $F \xrightarrow{M} F'$. Let N be a generic nilpotent matrix strictly compatible with both F and F'. Then

$$\mathrm{P}(M) = \mathrm{JF}^{\top}(N; F) \quad \textit{and} \quad \mathrm{Q}(M) = \mathrm{JF}^{\top}(N; F').$$

- If $F \xrightarrow{w} F'$, then $F' \xrightarrow{w^{-1}} F$. This implies the symmetry

$$\mathrm{P}(w^{-1}) = \mathrm{Q}(w).$$

- What happens when $\Bbbk$ is a *finite* field, and we consider *all* choices of N (not necessarily generic)?

q-Whittaker functions

- Define $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$.

Definition (q-Whittaker function)

$$W_\lambda(x_1, x_2, \ldots; q) := \sum_T \mathrm{wt}_q(T)\mathbf{x}^T,$$

where the sum is over all semistandard tableaux T of shape λ.

- $W_\lambda(\mathbf{x}; q)$ is symmetric in the variables x_i, and specializes to $s_\lambda(\mathrm{x})$ when $q = 0$. We obtain the $\mathfrak{gl}_n$-*Whittaker functions* as a certain $q \to 1$ limit.

- e.g. $T = $

1	2	4
3	5	7
6		

$\mathrm{wt}_q(T) = [1]_q[2]_q[1]_q[2]_q[2]_q[1]_q[2]_q = (1+q)^4$

- We have the following specializations:

$$W_\lambda(\mathbf{x}; q) = P_\lambda(\mathbf{x}; q, 0) = q^{\deg(\widetilde{H}_\lambda)}\omega(\widetilde{H}_\lambda(\mathbf{x}; 1/q, 0)), \quad W_\lambda(\mathbf{x}; 1) = e_{\lambda^\top}(\mathbf{x}).$$

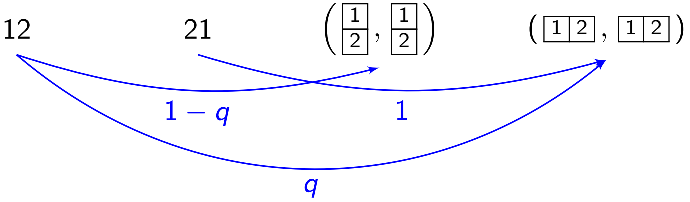
q-Cauchy identity

Theorem (Macdonald (1995))

$$\prod_{i,j\geq 1}\prod_{d\geq 0}\frac{1}{1-x_iy_jq^d}=\sum_{\lambda}\frac{(1-q)^{-\lambda_1}}{\prod_{i\geq 1}[\lambda_i-\lambda_{i+1}]_q!}W_\lambda(\mathbf{x};q)W_\lambda(\mathbf{y};q)$$

- This gives the *partition function* for the *q-Whittaker processes*, a special case of the *Macdonald processes* of Borodin and Corwin (2014).
- e.g. Taking the coefficient of $x_1x_2y_1y_2$ on each side gives

$$(1-q)^{-2}\ +\ (1-q)^{-2}\ =\ (1-q)^{-1}\ +\ (1-q)^{-2}(1+q)$$



q-Burge correspondence

- e.g. $w = 12$ $\quad N = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad (a \in \mathbb{F}_{1/q})$

$\mathrm{P}(w)$:	$\begin{array}{cc} & 1 \\ 1 & [\,0\,] \end{array}$	$\begin{array}{ccc} & 1 & 2 \\ 1 & 0 & a \\ 2 & 0 & 0 \end{array}$	$\mathrm{Q}(w)$: $\begin{array}{cc} & 1 \\ 1 & [\,0\,] \end{array}$	$\begin{array}{ccc} & 1 & 2 \\ 1 & 0 & a \\ 2 & 0 & 0 \end{array}$	
$a \neq 0$:	$\begin{array}{\|c\|}\hline \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline\end{array}$	$\mathbb{P} = 1 - q$
$a = 0$:	$\begin{array}{\|c\|}\hline \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline\end{array}$	$\mathbb{P} = q$

- e.g. $w = 21$ $\quad N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\mathrm{P}(w)$:	$\begin{array}{cc} & 1 \\ 1 & [\,0\,] \end{array}$	$\begin{array}{ccc} & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{array}$	$\mathrm{Q}(w)$: $\begin{array}{cc} & 2 \\ 2 & [\,0\,] \end{array}$	$\begin{array}{ccc} & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{array}$	
	$\begin{array}{\|c\|}\hline \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline\end{array}$	$\mathbb{P} = 1$

q-Burge correspondence

- Let $1/q$ be a prime power, and fix partial flags $F \xrightarrow{M} F'$ over $\mathbb{F}_{1/q}$. Let N denote a uniformly random nilpotent matrix strictly compatible with both F and F'. For semistandard tableaux T and T' of the same shape, define

$$\mathrm{p}_M(T, T') := \mathbb{P}[\mathrm{JF}^\top(N; F) = T \text{ and } \mathrm{JF}^\top(N; F') = T'].$$

(This definition depends only on M, not on the choice of (F, F').)

Theorem (Karp, Thomas (2021+))

(i) The maps $\mathrm{p}_M(\cdot, \cdot)$ define a probabilistic bijection proving the Cauchy identity for q-Whittaker functions, called the q-Burge correspondence.

(ii) As $q \to 0$, the q-Burge correspondence converges to the deterministic Burge correspondence.

- It is an open problem to determine if $\mathrm{p}_M(T, T')$ is a polynomial in q.
- A different probabilistic bijection was given by Matveev and Petrov (2017), using the *q-row insertion* of Borodin and Petrov (2016).

Proof outline

Theorem (Karp, Thomas (2021+))

Fix a nilpotent matrix N over $\mathbb{F}_{1/q}$ with Jordan type λ. The coefficient of $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ in $W_\lambda(\mathbf{x};q)$ equals $q^{\sum_i \binom{\lambda_i}{2} - \binom{\alpha_i}{2}}$ times the number of partial flags F strictly compatible with N satisfying

$$\dim(F_i) = \alpha_1 + \cdots + \alpha_i \quad \textit{for all } i.$$

- e.g. $\lambda =$ , $N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then the coefficient of $x_1 x_2$ in $W_\lambda(\mathbf{x};q)$ is

$$q^1 \cdot \#(\text{complete flags in } \mathbb{F}_{1/q}^2) = q(1 + 1/q) = q + 1.$$

- This is similar to a formula for the *modified Hall–Littlewood functions* $\widetilde{H}_\lambda(\mathbf{x};q,0)$ in terms of *weakly* compatible flags over $\mathbb{F}_q$.
- To prove that the q-Burge correspondence works, we double count

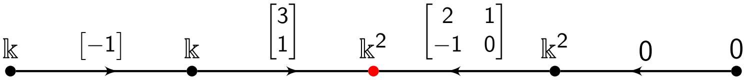
$$\{(F, F', N) : F \xrightarrow{M} F',\ \mathrm{JF}^\top(N;F) = T,\ \mathrm{JF}^\top(N;F') = T'\}.$$

Quiver representations and the preprojective algebra

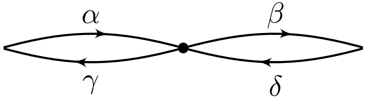
- Consider a path quiver with a unique sink:

$Q =$ 

- A *representation* V of Q is an assignment of a vector space to each vertex and a linear map to each arrow, e.g.,

$V =$ 

- We will only consider V where every linear map is injective. Isomorphism classes of such V are indexed by nonnegative-integer matrices M.
- We now decorate V with a linear map for the reverse of each arrow, such that a relation holds for every vertex:

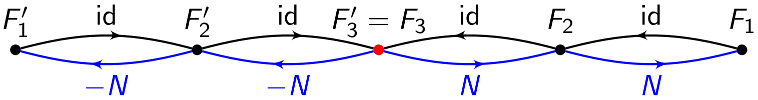


$$\alpha \circ \gamma \pm \delta \circ \beta = 0$$

This defines a module $V^\sharp$ over the *preprojective algebra* of Q.

Socle filtration

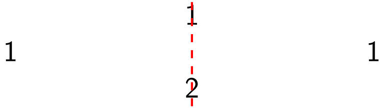
- Up to isomorphism, $V^\sharp$ is given (non-uniquely) by a triple (F, F', N):

$V^\sharp =$ 

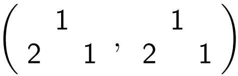
- The *socle filtration* of $V^\sharp$ corresponds precisely to the pair of tableaux

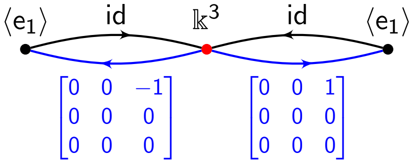
$$(T, T') = (\mathrm{JF}^\top(N; F), \mathrm{JF}^\top(N; F')).$$

- e.g.



$\longleftrightarrow$



$V^\sharp =$ 

$$\longleftrightarrow \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array} \right)$$

Counting isomorphism classes

- The q-Burge correspondence gives enumerative results such as:

Theorem (Karp, Thomas (2021+))

Let (T, T') be a pair of semistandard tableaux of shape λ, and let $\mathbf{d}$ be a dimension vector of Q. Then

$$\sum_{V^\sharp} \frac{1}{|\mathrm{Aut}(V^\sharp)|} = \frac{q^{c(\mathbf{d})}(1-q)^{-\lambda_1}}{\prod_{i\geq 1}[\lambda_i - \lambda_{i+1}]_q!} \, \mathrm{wt}_q(T)\, \mathrm{wt}_q(T'),$$

where the sum is over all $V^\sharp$ over $\mathbb{F}_{1/q}$ up to isomorphism, with dimension vector $\mathbf{d}$ and socle filtration corresponding to (T, T').

- The description of $W_\lambda(\mathbf{x}; q)$ in terms of partial flags can be rephrased as enumerative statements for certain *Nakajima quiver varieties* over $\mathbb{F}_{1/q}$.

Thank you!