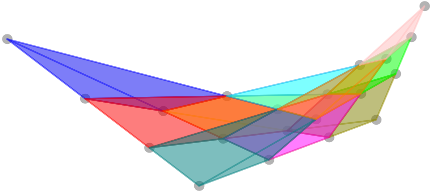
Geometry of the amplituhedron

Slides available at `www-personal.umich.edu/~snkarp`



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`arXiv:1608.08288` (joint with Lauren Williams)
`arXiv:1708.09525` (joint with Lauren Williams and Yan Zhang)

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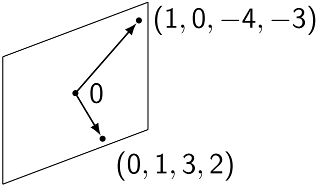
The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces V of $\mathbb{R}^n$.

$V :=$ 

$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

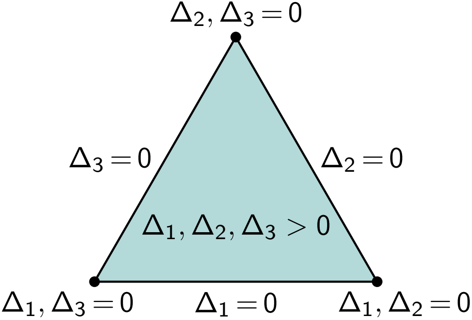
$$\Delta_{12}(V) = 1 \qquad \Delta_{13}(V) = 3 \qquad \Delta_{14}(V) = 2$$
$$\Delta_{23}(V) = 4 \qquad \Delta_{24}(V) = 3 \qquad \Delta_{34}(V) = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \ldots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call V *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all such V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.

The 'faces' of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition due to Rietsch (1999) and Postnikov (2007). Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

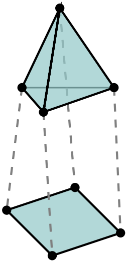
$\mathrm{Gr}_{1,3}^{\geq 0} = \mathbb{P}_{\geq 0}^{2} \cong$



- $\mathrm{Gr}_{1,n}^{\geq 0}$ is an $(n-1)$-dimensional simplex in $\mathbb{P}^{n-1}$. So, one can think of $\mathrm{Gr}_{k,n}^{\geq 0}$ as a generalization of a simplex into the Grassmannian.

Amplituhedra and Grassmann polytopes

- By definition, a polytope is the image of a simplex under an affine map:



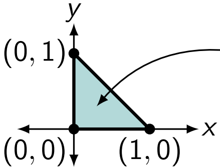
$$\begin{tikzcd}
\text{simplex in } \mathbb{P}^{n-1} \arrow[r, dashed] \arrow[d, mapsto, "\substack{\text{linear map} \\ \mathbb{R}^n \to \mathbb{R}^{m+1}}"] & \mathrm{Gr}_{k,n}^{\geq 0} \arrow[d, mapsto, "\substack{\text{linear map} \\ \mathbb{R}^n \to \mathbb{R}^{k+m}}"] \\
\text{polytope in } \mathbb{P}^m \arrow[r, dashed] & \substack{\textit{Grassmann polytope} \\ \text{in } \mathrm{Gr}_{k,k+m}}
\end{tikzcd}$$

A *Grassmann polytope* is the image of a map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

- When the matrix Z has positive maximal minors, the Grassmann polytope is called an *amplituhedron*, denoted $\mathcal{A}_{n,k,m}(Z)$. Amplituhedra generalize cyclic polytopes ($k = 1$) and totally nonnegative Grassmannians ($k + m = n$). They were introduced by Arkani-Hamed and Trnka (2014), and inspired Lam (2015) to define Grassmann polytopes.

Positive geometries and canonical forms

- Arkani-Hamed, Bai, Lam (2017): a *positive geometry* is a space equipped with a canonical differential form, which has logarithmic singularities at the boundaries of the space. Examples include convex polytopes:



$$\frac{dxdy}{xy(1-x-y)}$$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ is a positive geometry. The canonical form of e.g. $\mathrm{Gr}_{2,4}^{\geq 0}$ is

$$\frac{dxdydzdw}{\Delta_{12}\Delta_{23}\Delta_{34}\Delta_{14}}, \text{ where } V = \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{bmatrix} \in \mathrm{Gr}_{2,4}\,.$$

It also equals (up to sign)

$$\frac{d\Delta_{23}}{\Delta_{23}}\frac{d\Delta_{34}}{\Delta_{34}}\frac{d\Delta_{14}}{\Delta_{14}}\frac{d\Delta_{13}}{\Delta_{13}}, \text{ where } \{\Delta_{12}=1, \Delta_{23}, \Delta_{34}, \Delta_{14}, \Delta_{13}\} \text{ is a cluster.}$$

Canonical forms in physics

- The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is conjecturally a positive geometry, whose canonical form for $m = 4$ is the tree-level scattering amplitude in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory.
- Other positive geometries with physically relevant canonical forms include *associahedra*, *Cayley polytopes*, and *cosmological polytopes*.

Problem

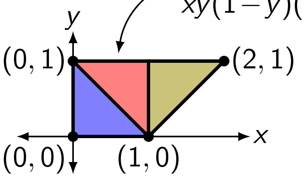
Let A be a cluster algebra. Consider the form

$$\omega := \pm \bigwedge_{i=1}^{n} \frac{dx_i}{x_i}, \text{ where } \{x_1, \ldots, x_n\} \text{ is a cluster of } A.$$

Note that ω does not depend (up to sign) on the choice of cluster.
(i) Is ω the canonical form of a positive geometry?
(ii) Is there a physical interpretation of ω?

Triangulations

- One way to find the canonical form of a positive geometry is by triangulating it into simpler pieces:



$$\frac{(1+y)dxdy}{xy(1-y)(1-x+y)} = \frac{dxdy}{xy(1-x-y)} + \frac{dxdy}{(1-x)(1-y)(x+y-1)} + \frac{dxdy}{(x-1)(1-y)(1-x+y)}$$

Conjecture (Arkani-Hamed, Trnka (2014))

The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is 'triangulated' by the images of certain $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, coming from the BCFW recursion.

Problem

Find a 'triangulation-independent' description of the amplituhedron form.

Sign variation

- As a simpler case, we first considered $\mathcal{A}_{n,k,m}(Z)$ for $m = 1$.
- For $v \in \mathbb{R}^n$, let $\mathrm{var}(v)$ be the number of sign changes in the sequence $(v_1, v_2, \dots, v_n)$, ignoring any zeros. Let $\overline{\mathrm{var}}(v)$ be the maximum number of sign changes we can get if we choose a sign for each zero component of v.

$$\mathrm{var}(1, -4, 0, -3, 6, 0, -1) = \mathrm{var}(1, -4, -3, 6, -1) = 3$$

$$\overline{\mathrm{var}}(1, -4, 0, -3, 6, 0, -1) = 5$$

Theorem (Gantmakher, Krein (1950))

Let $V \in \mathrm{Gr}_{k,n}$. The following are equivalent:
(i) V is totally nonnegative;
(ii) $\mathrm{var}(v) \le k - 1$ for all $v \in V \setminus \{0\}$;
(iii) $\overline{\mathrm{var}}(w) \ge k$ for all $w \in V^{\perp} \setminus \{0\}$.

- e.g. $\begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\ge 0}$.
- We can characterize *totally positive* V by switching var and $\overline{\mathrm{var}}$ above.

Sign variation and the amplituhedron

- Recall: $\mathcal{A}_{n,k,m}(Z)$ is the image of $\mathrm{Gr}_{k,n}^{\geq 0}$ under the $(k+m)\times n$ matrix Z.

Lemma (Karp, Williams)

Let $W \in \mathrm{Gr}_{k+m,n}$ denote the subspace spanned by the rows of Z.
(i) $\mathcal{A}_{n,k,m}(Z) \cong \mathcal{B}_{n,k,m}(W) := \{V^{\perp} \cap W : V \in \mathrm{Gr}_{k,n}^{\geq 0}\} \subseteq \mathrm{Gr}_m(W)$.
(ii) $\mathcal{B}_{n,k,m}(W) \subseteq \{X \in \mathrm{Gr}_m(W) : k \leq \overline{\mathrm{var}}(v) \leq k+m-1 \text{ for } v \in X \setminus \{0\}\}$.

Problem

Does equality hold in (ii) above?

- If so, we can translate the sign-variation conditions on vectors in X into conditions on sequences of Plücker coordinates of X by results of Karp (2017). Simpler such (conjectural) conditions were discovered independently by Arkani-Hamed, Thomas, and Trnka (2018).
- We showed that equality does hold in (ii) when $m = 1$:

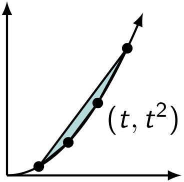
$$\mathcal{B}_{n,k,1}(W) = \{w \in \mathbb{P}(W) : \overline{\mathrm{var}}(w) = k\} \subseteq \mathbb{P}(W).$$

Cyclic hyperplane arrangements

- A *cyclic polytope* is a polytope (up to combinatorial equivalence) whose vertices line on the *moment curve*

$$f(t) := (t, t^2, \ldots, t^m) \text{ in } \mathbb{R}^m \quad (t > 0).$$

- e.g. $m = 2$



- The key property is that for any $t_1 < \cdots < t_n$, the $(m+1) \times n$ matrix

$$\begin{bmatrix} 1 & \cdots & 1 \\ f(t_1) & \cdots & f(t_n) \end{bmatrix} \text{ has positive maximal minors.}$$

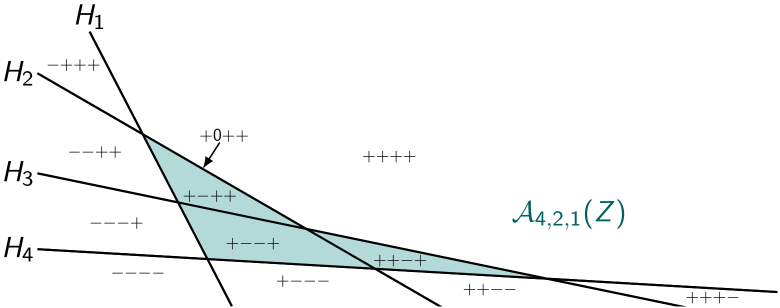
More generally, every $k = 1$ amplituhedron $\mathcal{A}_{n,1,m}(Z)$ is a cyclic polytope.

- A *cyclic hyperplane arrangement* consists of hyperplanes of the form

$$tx_1 + t^2x_2 + \cdots + t^kx_k + 1 = 0 \text{ in } \mathbb{R}^k \quad (t > 0).$$

The $m = 1$ amplituhedron

- e.g.



Theorem (Karp, Williams)

(i) $\mathcal{A}_{n,k,1}(Z)$ is isomorphic to the complex of bounded faces of a cyclic hyperplane arrangement of n hyperplanes in $\mathbb{R}^k$.
(ii) $\mathcal{A}_{n,k,1}(Z)$ is isomorphic to a subcomplex of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$.
(iii) $\mathcal{A}_{n,k,1}(Z)$ is homeomorphic to a closed ball of dimension k.

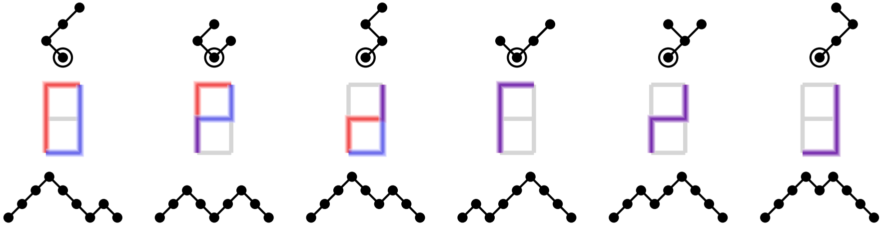
- Recall: $\mathcal{A}_{n,k,1}(Z) \cong \mathcal{B}_{n,k,1}(W) = \{w \in \mathbb{P}(W) : \overline{\mathrm{var}}(w) = k\} \subseteq \mathbb{P}(W)$.

BCFW triangulation for $m = 4$

Conjecture (Arkani-Hamed, Trnka (2014))

The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is 'triangulated' by the images of certain $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, coming from the BCFW recursion.

- The number of top-dimensional cells in a BCFW triangulation is the *Narayana number* $N_{n-3,k+1} := \frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$.
- e.g. For $n = 7$, $k = 2$, we have $N_{7-3,2+1} = 6$:



- $k = 1$, m even: every triangulation of $\mathcal{A}_{n,1,m}(Z)$ has $\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$ top cells.
- $m = 2$: there is a nice triangulation of $\mathcal{A}_{n,k,2}(Z)$ with $\binom{n-2}{k}$ top cells.

Number of cells in a triangulation

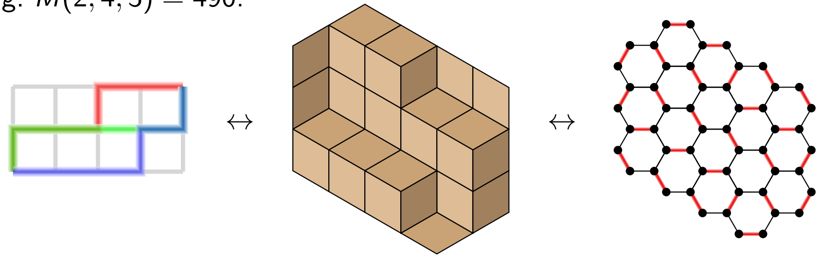
- Define the *MacMahon number*

$$M(a,b,c) := \prod_{p=1}^{a}\prod_{q=1}^{b}\prod_{r=1}^{c} \frac{p+q+r-1}{p+q+r-2}.$$

Conjecture (Karp, Williams, Zhang)

For m even, there exists a cell decomposition of $\mathcal{A}_{n,k,m}(Z)$ with $M(k, n-k-m, \frac{m}{2})$ top-dimensional cells.

- $M(a,b,c)$ is the number of *plane partitions* inside an $a \times b \times c$ box.
- e.g. $M(2,4,3) = 490$:



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Problem

Interpret properties of plane partitions in terms of amplituhedra.

- The $k \leftrightarrow n-k-m$ symmetry comes (for $m=4$) from *parity* of the scattering amplitude. Galashin and Lam (2018) showed that the *stacked twist map* interchanges triangulations of $\mathcal{A}_{n,k,m}(Z)$ and $\mathcal{A}_{n,n-k-m,m}(Z')$.

Problem

Explain the conjectural symmetry for amplituhedra between k and $\frac{m}{2}$.