Positive geometries - relatively new branch of math w/ connections to physics

$P \subset \mathbb{R}^d$ full-dim'l polytope

Defn (dual volume function):

$$\mathrm{Vol}_z^{\vee}(P) := \mathrm{Vol}\left((P-z)^{\vee}\right)$$

$z = (z_1, z_2, \dots, z_n)$

where $P^{\vee} = \{ \underline{x} \in \mathbb{R}^d \mid \langle \underline{x}, \underline{y} \rangle \geq -1 \ \forall \underline{y} \in P \}$ Polar Polytope

Note: We use normalized volume

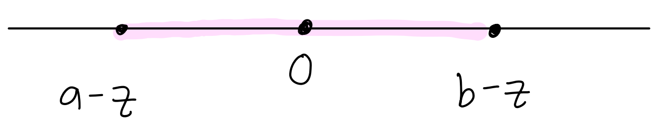
volume of cube $= d!$

volume of simplex $= 1$

EX: $P = [a, b] \subset \mathbb{R}$

P is highlighted segment

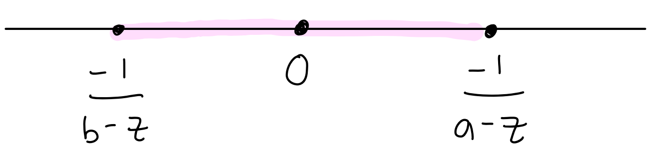
$P - z = [a-z, b-z]$



Choose z s.t. 0 is in the interior of P

$$P^{\vee} = \left\{ x \in \mathbb{R} \;\middle|\; \begin{array}{l} x \cdot (a-z) \geq -1 \\ x \cdot (b-z) \geq -1 \end{array} \right\} \Longrightarrow \frac{-1}{b-z} \leq x \leq \frac{-1}{a-z}$$

So,



, $(P-z)^{\vee} = \left[\frac{1}{z-b}, \frac{1}{z-a} \right]$

Thus, $\mathrm{Vol}^{\vee}_{z}(P) = \dfrac{1}{z-a} - \dfrac{1}{z-b}$

Defn (canonical form):

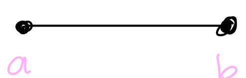
$$\Omega(P) := \mathrm{Vol}^{\vee}_{z}(P)\, dz_1\, dz_2 \cdots dz_d$$

↳ Depends on polytope P and orientation

Thm: $\Omega(P)$ is the unique rational top form on $\mathbb{P}^d$ satisfying the recursion $\mathrm{Res}_F\, \Omega(P) = \Omega(F)$ for all facets $F \subset P$ and has no further poles.

↳ Initial condition: $\Omega(pt) = \pm 1$

Ex: $P = [a, b]$, P has 2 1-dim'l facets



- $\mathrm{Res}_{z=a}\left(\dfrac{1}{z-a} - \dfrac{1}{z-b}\, dz\right) = \dfrac{1}{2\pi i}\displaystyle\oint_{|z-a|=\varepsilon} \left(\dfrac{1}{z-a} - \dfrac{1}{z-b}\, dz\right)$

  $= 1$

- $\mathrm{Res}_{z=b}\left(\dfrac{1}{z-a} - \dfrac{1}{z-b}\, dz\right) = \dfrac{1}{2\pi i}\displaystyle\oint_{|z-b|=\varepsilon} \left(\dfrac{1}{z-a} - \dfrac{1}{z-b}\, dz\right)$

  $= -1$

Thm says "polytopes are positive geometries"

↳ uniquely determined by this recursion

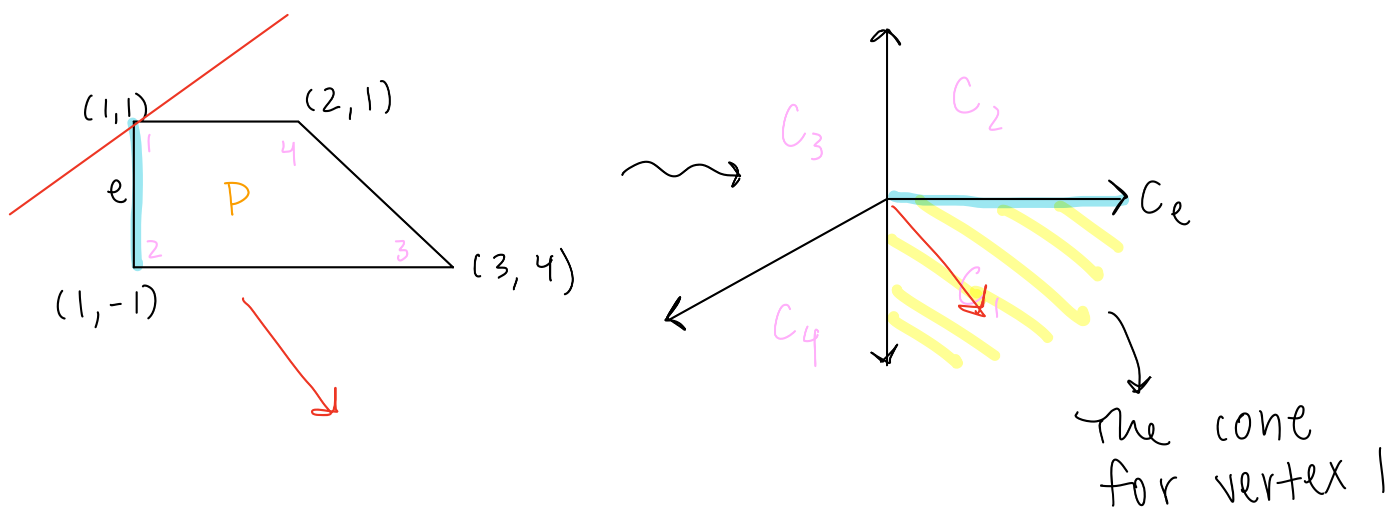
For each face F, $C_F := \{ \underline{v} \in \mathbb{R}^d \mid \langle \underline{v}, y \rangle = \min_{\underline{x} \in P} \langle \underline{v}, \underline{x} \rangle \ \forall y \in F \}$

is the closed cone associated to a face F.

The collection of $\{ C_F \mid F \text{ face} \}$ is the normal fan $N(P)$.

Define $h_P : \mathbb{R}^d \longrightarrow \mathbb{R}$, $h_P(\underline{v}) := -\min_{\underline{x} \in P} \langle \underline{v}, \underline{x} \rangle$ as the support function.

Ex:

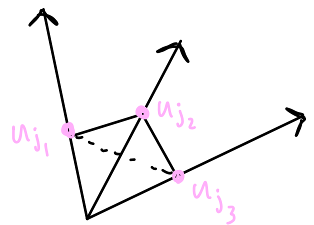


Element is in C_1 if the vector is perpendicular to the line (or plane) passing through vertex 1 (indicated w/ red lines above)

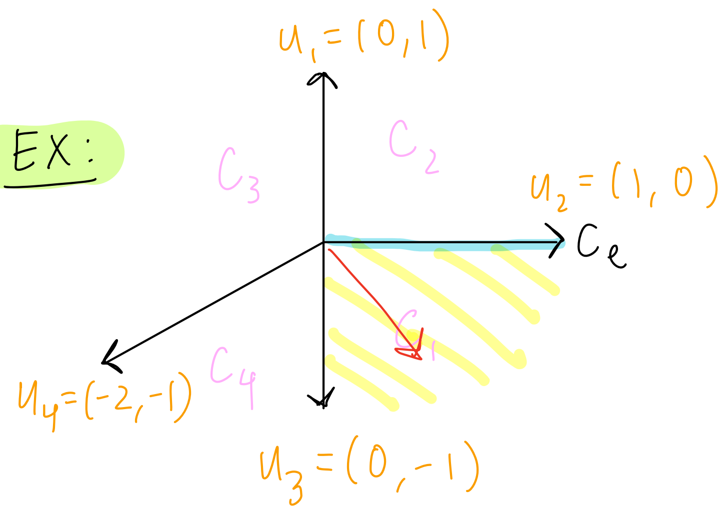
Thm: Let T be a triangulation of $N(P)$ with $u_1, \dots, u_m$ (integral) generators of rays. Then

$$\mathrm{Vol}^{\vee}_{z}(P) = \sum_{\underbrace{C=\mathrm{span}(u_{j_1},\dots,u_{j_d})}_{\text{max cone}}} \frac{\left|\det|u_{j_1},\dots,u_{j_d}|\right|}{\underbrace{h_{P-z}(u_{j_1})}_{\text{linear in } z}\cdots h_{P-z}(u_{j_d})}$$

↳ **EX:**



EX:



$$\text{so, } \mathrm{vol}^{\vee}_{z}(P) = \frac{1}{(1+z_2)(-1+z_1)} + \frac{1}{(-1+z_1)(1-z_2)} + \frac{2}{(1-z_2)(5-2z_1-z_2)}$$

$$+ \frac{2}{(5-2z_1-z_2)(1+z_2)} = \frac{6-2z_1}{(1+z_2)(\;)(\;)(\;)}$$

Observe: $h_P(u_4)=5$

Note: The numerator will be linear in z

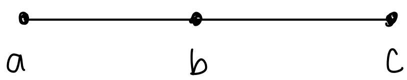
Thm: $P \longmapsto \mathrm{Vol}^{\vee}_{z}(P)$ is valuative.

Let $[P]$ = character function of P.

Suppose $\sum_{i=1}^{r} \alpha_i [P_i] = 0$. Then $\sum_{i=1}^{r} \alpha_i \operatorname{Vol}_z^\vee(P) = 0$ where if

$\dim(P) < d$, $\operatorname{Vol}_z^\vee(P) = 0$.

Ex:



$$\big[[a,b]\big] + \big[[b,c]\big] - \big[[a,c]\big]$$

$$[b] = 0$$

$$\Rightarrow \operatorname{Vol}_z^\vee([a,b]) + \operatorname{Vol}_z^\vee([b,z]) = \operatorname{Vol}_z^\vee([a,c])$$

$\operatorname{Vol}_z^\vee(P)$ is a rational function of degree $-(d+1)$

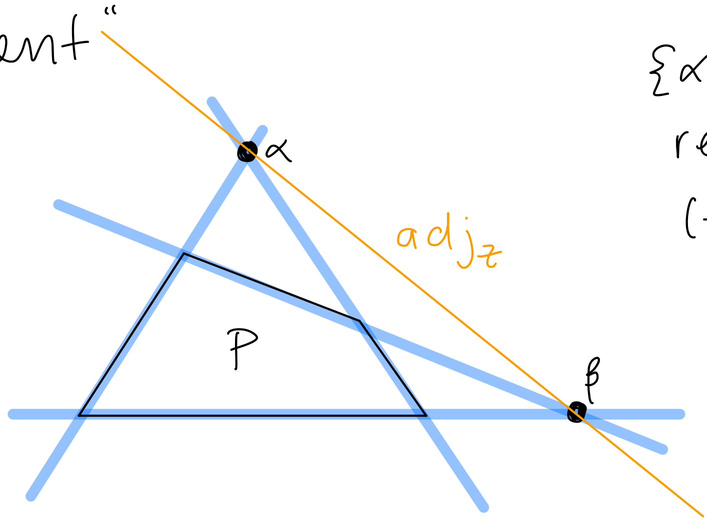
since we use polar function

adjoint

$$\rightsquigarrow \operatorname{Vol}_z^\vee(P) = \frac{\operatorname{adj}_z(P^\vee)}{\prod_{\text{facets}} (\text{linear})} \rightarrow \deg = \#\text{facets} - d - 1$$

Thm: $\operatorname{adj}_z(P^\vee)$ vanishes on the "residual arrangement"

Ex:



$\{\alpha, \beta\}$ is the residual arrangement (these are the points that give us more polytopes when intersecting hypersurfaces)

The numerator of $\mathrm{vol}_z^\vee(P)$ is called the adjoint hypersurface

(Kohn-Ranestad)

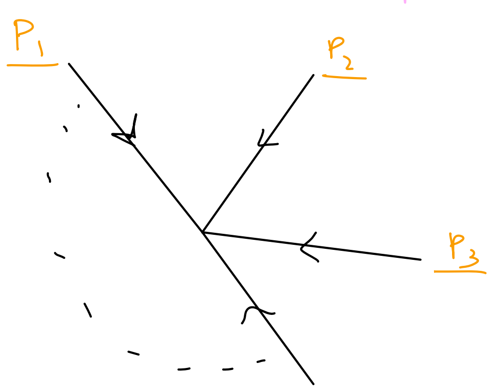
Thm: If P is simple and generic, then $\mathrm{adj}_z(P^\vee)$ is determined up to scalar by this vanishing.

In particle physics, you can calculate "scattering amplitude" $A_n(\underbrace{\underline{p}_1, \ldots}_{4n \text{ variables}})$

↳ $\mathrm{Prob}(\text{event}) \propto \int_{\text{detection}} |A_n|^2$

$$A_n = \sum_{L=0}^{\infty} \sum_{\Gamma} \int_{\mathbb{R}^{4L}} (\text{rational form}) \quad , \text{ loops } h'(P) = L$$

depends on the momenta of the particles involved $\rightsquigarrow$ $\underline{p}_i \in \mathbb{R}^{1,3}$



This is how we can think of the particles moving

Ex: $A_n = \sum_{L=0}^{\infty} A_n^{(L)}$

$A_n^{(0)}$ is the "tree amplitude"

$A_n^{(0)} = \sum_{\text{graphs}} (\text{rational functions})$

↳ came about from conjecture (~80s) $\dfrac{\langle ij \rangle^4}{\langle 12\rangle\langle 23\rangle \dots \langle 1n\rangle}$

We note the relationship

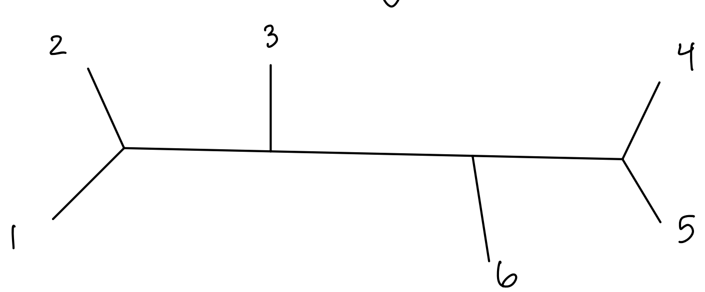
- amplitude function
- canonical form

$\longleftrightarrow$

- amplitahedron
- geometry (polytope)
- positive geometry

Ex: Planar φ^3 theory

Feynman diagrams = planar cubic trees

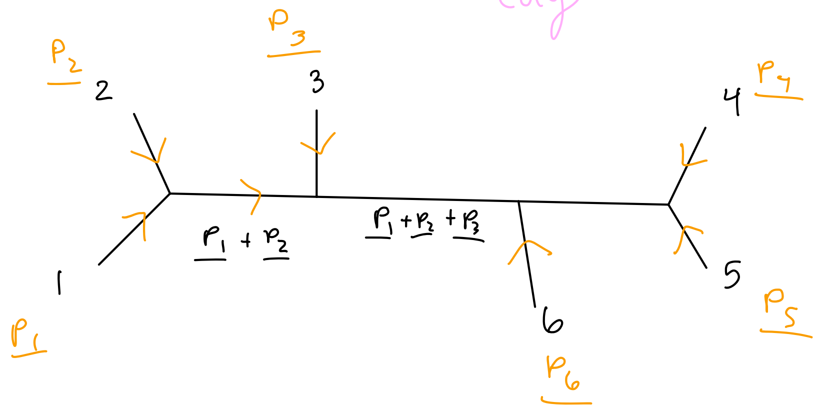


The number of these trees is C_{n-2} (Catalan number $n-2$)

$\exists$ polytope "associahedron" with vertices labeled by these Feynman diagrams

EX: $A_n^{\varphi^3,\text{tree}} = \sum_T \prod_e \frac{1}{X_e}$ ← (momentum along edge)^2

↑ internal edges



Thm: There is an associahedron $Ass_{n-3}(\underline{p_1}, \dots, \underline{p_n})$

s.t. $\text{Vol}_z^\vee(Ass_{n-3}) = A_n^{\varphi^3,\text{tree}}$

↳ $A_n(\underline{p})$ is a function of $s_{ij} = \underline{p_i} \cdot \underline{p_j}$