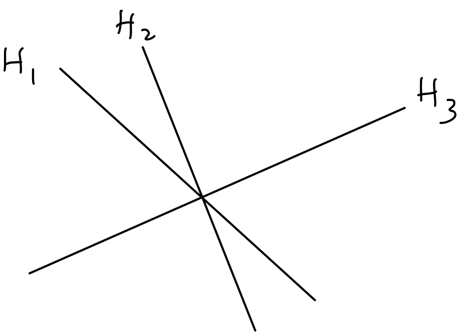
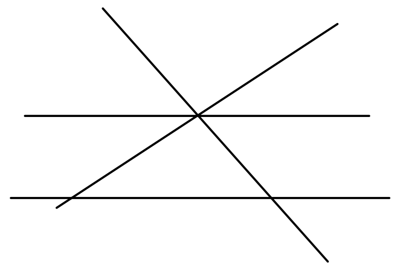
$\mathcal{A} = \{H_1, H_2, \dots, H_n\}$ → hyperplane arrangement

EX:

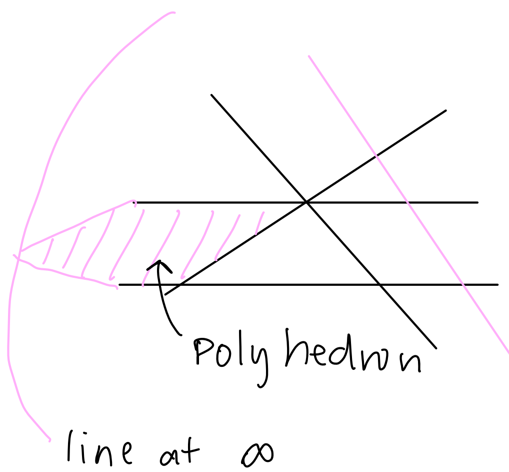


$\subset \mathbb{R}^2$



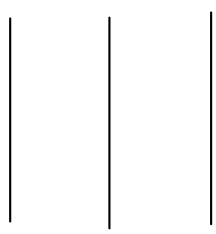
Central: All H_i's pass through origin
↳ written $\mathcal{A}$
↳ $\mathcal{A} = \{H_e \mid e \in E\}$ ↓ ground set

Affine: All H_i's do not pass through origin → written $\bar{\mathcal{A}}$
↳ 4 polytopes in this picture



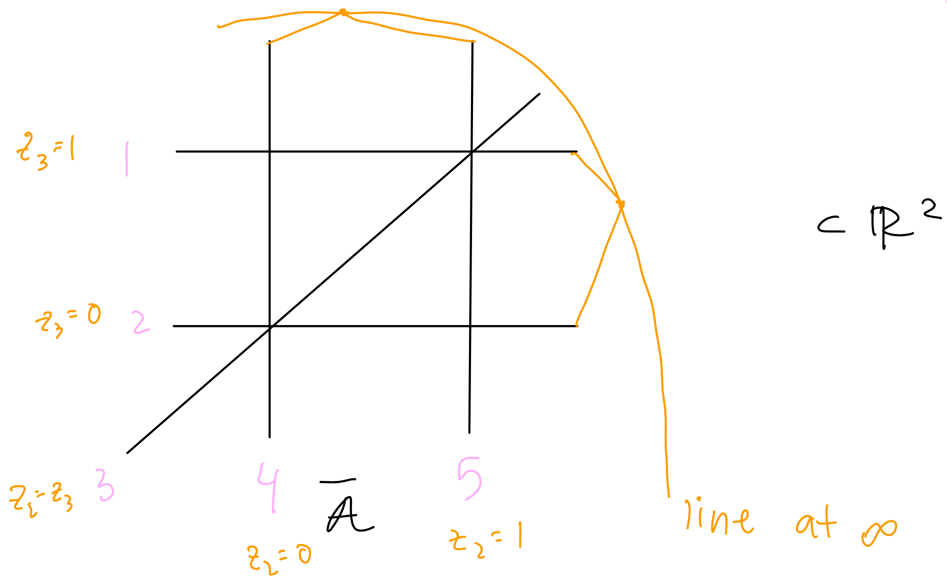
An arrangement is **essential** if the normals span $\mathbb{R}^d$

↳ **EX:**



$\subset \mathbb{R}^2$ is <u>not</u> essential

Ex: $\mathbb{C}^2 \setminus \bar{\mathcal{A}} = \bar{\mathcal{U}} = \mathcal{M}_{0,5}(\mathbb{C})$ Hyperplane arrangement complement



Note:

$\mathcal{M}_{0,5} = \{(z_1, z_2, z_3, z_4, z_5) \in (\mathbb{P}^1)^5$ s.t. $z_i \neq z_j, i \neq j\}$

$\hookrightarrow PGL(2)$

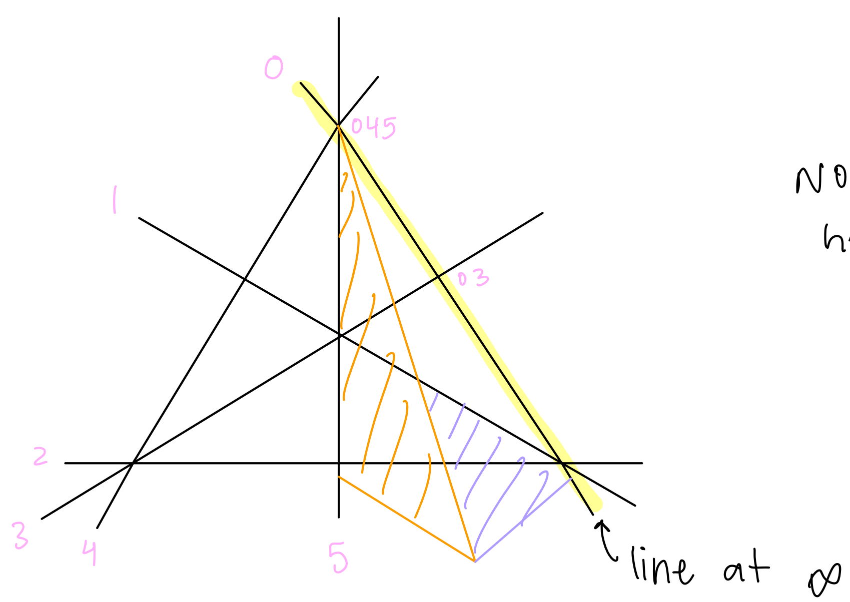
our ex: $(0, z_2, z_3, 1, \infty)$

$z_2 \neq 0, 1, \infty$

$z_2 \neq z_3$

$z_3 \neq 0, 1, \infty$

Now we want to turn this to a picture in $\mathbb{P}^2$



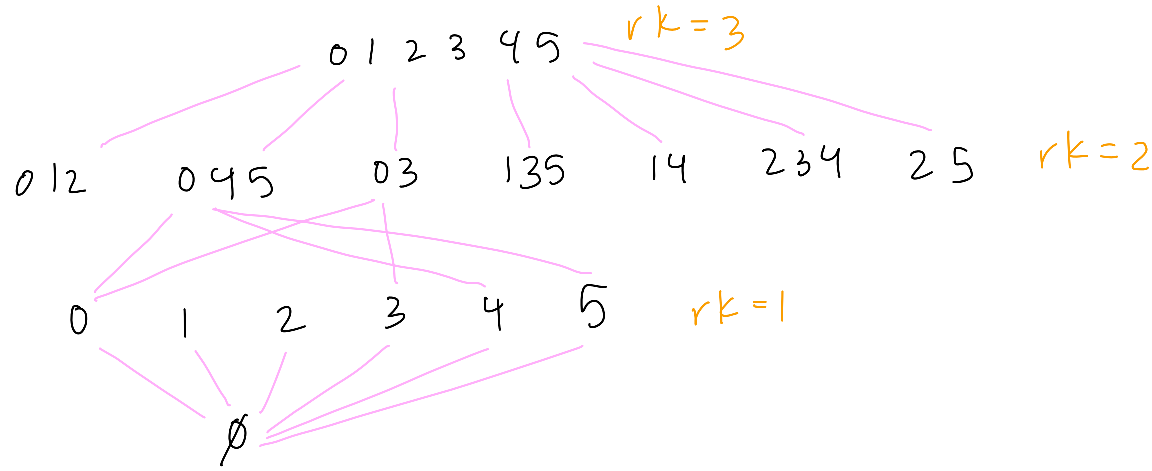
Now we have 6 hyperplanes

To make this central, we add 2 planes $\rightsquigarrow$ orange and purple spaces in above picture

We can store lots of information in the lattice of flats of the matroid associated to the hyperplane arrangement (affine)

Defn: $L(\mathcal{A})$ (pretend it's more symmetric and nicer working...)

This is a poset ordered by inclusion



We define a matroid by its lattice of flats $L(\mathcal{A})$

Goal:
- characteristic poly.: $\chi_M(t) := \sum_{F \in L} t^{r - rk(F)} \mu(\emptyset, F)$
- Orlik-Solomon algebra: $A(M)$

If $\mathcal{A}$ is define $/\mathbb{Z}$, $\chi_{\mathcal{A}}(q) = \#\{U(\mathbb{F}_q)\}$

Ex: For $M_{0,5}$

- $(z_2, z_3) \in (\mathbb{F}_q)^2 \rightsquigarrow \chi_{M_{0,5}}(t) = (t-2)(t-3)$

  $z_2 \neq 0, 1$

  $z_3 \neq 0, 1$

  $z_2 \neq z_3$

- $(z_2, z_3) \in (\mathbb{P}(\mathbb{F}_q))^2 \rightsquigarrow \chi_{M_{0,5}}(t) = (t-2)(t-3)$

  $z_2 \neq 0, 1, \infty$

  $z_3 \neq 0, 1, \infty$

  $z_2 \neq z_3$

The reduced characteristic poly. is defined by

$$\overline{\chi}_M(t) = \chi_M(t) / (t-1)$$

Note: We use $\overline{\chi_M}(t)$ for $\bar{A}$ and $\chi_M(t)$ for A

Thm: (Zaslavsky)

$|\bar{\chi}(1)| = \#$ of bounded chambers

$|\bar{\chi}(-1)| = \#$ of chambers

EX: From our previous ex, $|\chi(1)| = 2$ (we can see on picture earlier)

$|\chi(-1)| = 12$

Defn: (Orlik-Solomon algebra)

$$A^{\bullet}(M) = \Lambda^{\bullet}(E) / (\partial e_S \mid S \subseteq E \text{ dependent})$$

- E is ground set
- S is dependent if $\operatorname{codim} \bigcap_{s \in S} H_s \neq |S|$
- $e_S := e_{s_1} \wedge e_{s_2} \wedge \dots \wedge e_{s_k}$ if $S = \{s_1, s_2, \dots, s_k\}$

$$- \partial(e_1 \wedge \dots \wedge e_k) := \sum_{i=1}^{k} (-1)^{k-i} e_1 \wedge \dots \wedge \hat{e_i} \wedge \dots \wedge e_k$$

Defn: (Reduced Orlik-Solomon algebra)

$$\overline{A}^\bullet(M) = \langle \overline{e} := e - e_0 \mid e \in E \setminus 0 \} \subset A^\bullet(M)$$

Prop: Hilbert series of $A^\bullet(M)$ is $t^r \chi_M(\frac{1}{t})$ up to sign (of coeff.)

rank

Prop: Hilbert series of $\overline{A}^\bullet(M)$ is $t^{r-1} \overline{\chi}_M(\frac{1}{t})$ up to sign (of coeff.)

Ex: $\overline{\chi}(t) = (t-2)(t-3) = t^2 - 5t + 6$

Claim:
$$\overline{A}^0 = 1$$
$$\overline{A}^1 = 5$$
$$\overline{A}^2 = 6 = \binom{5}{2} - 4$$

relations in degree 2

some ideal

$$A^\bullet(M) = \Lambda^\bullet(e_0, e_1, \dots, e_5) / (\cdots)$$

∂e_{045}
∂e_{012}
∂e_{135}
∂e_{234}

These are the dependent sets

↳ degree 2 relations

$$\overline{A}^\bullet(M) = \langle e_1 - e_0, e_2 - e_0, \dots, e_5 - e_0 \rangle$$

$$\Rightarrow \dim(\Lambda^0(E)) = 1$$

$\dim(\Lambda^1(E)) = 6$

$\dim(\Lambda^2(E)) = \binom{6}{2}$

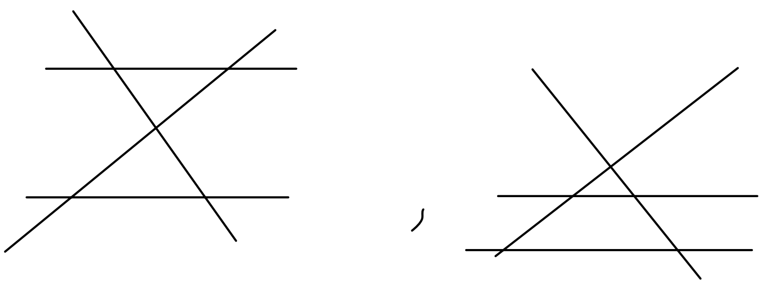
Thm: (Orlik-Solomon)

(1) $H^*(\mathcal{U}(\mathbb{C})) \cong A^\bullet(M)$, $\mathcal{U} := \mathbb{C}^r \setminus \mathcal{A}$

(2) $H^*(\overline{\mathcal{U}}(\mathbb{C})) \cong \overline{A}^\bullet(M)$, $\overline{\mathcal{U}} := \mathbb{C}^d \setminus \overline{\mathcal{A}}$

↳ These rings <u>only</u> depend on the matroid M

EX:



These 2 arrangements have the same cohomology in the complement

Maps for our thm: $H_e = \{f_e = 0\}$

(1) $e \longmapsto d\log(f_e)$

$d\log(f_e) = \frac{df_e}{f_e}$

(2) $\overline{e} \longmapsto d\log\left(\frac{f_e}{f_0}\right)$

EX: From our $M_{0,5}$ example

$H^*(M_{0,5}) = \langle d\log(z_2), d\log(z_2 - 1), d\log(z_3), \ldots, d\log(z_2 - z_3) \ldots \rangle$

$d\log(z_2 - 1) = \frac{dz_2}{z_2 - 1}$

Fact: For any chamber C, $\Omega(C) \in \bar{A}^d(M)$ top degree

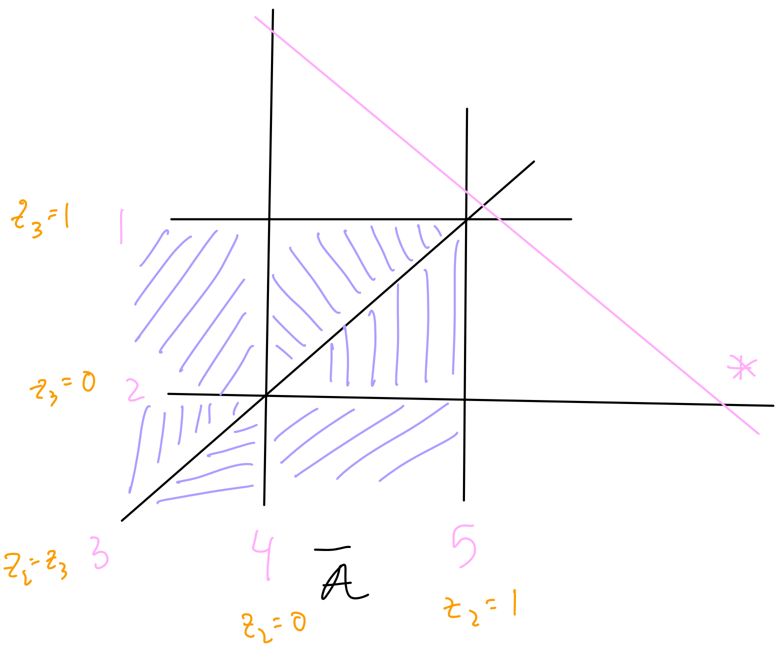
↳ There are 12 degree 2 canonical forms for $M_{0,5}$

Thm: (Yoshinaga, Eur, Lam)

$\{\Omega(C) \mid C \text{ is bounded w.r.t. a generic ext.}\}$

form a basis of $\bar{A}^d(M)$

EX:



Bounded chambers are chambers that don't intersect (*)

↳ purple chambers to the left

Fact: $\bar{A}^\bullet(M)$ has a classical "no-broken circuit" (NBC) basis

Aomoto complex:

$\omega \in \bar{A}^1(M)$

$= \sum_{e \in E \setminus 0} a_e \bar{e}, \quad a_e \in \mathbb{C}$

$$\rightsquigarrow \quad \bar{A}^0(M) \xrightarrow{\wedge \omega} \bar{A}^1(M) \xrightarrow{\wedge \omega} \bar{A}^2(M) \longrightarrow \cdots =: (\bar{A}^\bullet(M), \omega)$$

Fact: For generic ω, $H^*((\bar{A}^\bullet(M), \omega))$ is concentrated in top d-degree

↳ Known: $\bar{A}^d(M) / \omega \bar{A}^{d-1}(M)$ is the top cohomology

$$\dim\left(\bar{A}^d(M) / \omega \bar{A}^{d-1}(M)\right) = |\chi_M(1)|$$

Thm: (Yoshinaga, Eur, Lam)

$\{\Omega(C) \mid C \text{ is bounded}\}$ form a basis of

$$\bar{A}^d(M) / \omega \bar{A}^{d-1}(M)$$

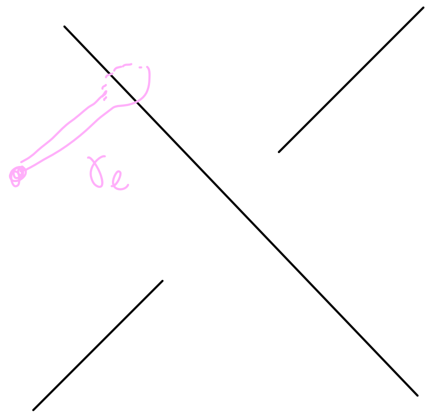
$\bar{U} = \mathbb{C}^d \setminus \mathcal{A}$

We can consider rank 1 "local systems" on $\bar{U}$.
These are classified up to isomorphism

by conjugacy classes of representations

$$\rho : \pi_1(\bar{U}) \longrightarrow GL_1 = \mathbb{C}^*$$

Ex:



There is only one relation to check:

$$\prod_e \rho(\gamma_e) = 1$$

Let $\rho(\gamma_e) =: b_e$. For a tuple $(b_e \mid e \in E)$ we get a local system $\mathcal{L}_b$.

Thm:

$$H^*(\bar{U}, \mathcal{L}_b) \cong H^*((\bar{A}^\bullet(M), \omega))$$

for generic b where $b_e = \exp(-2\pi i a_e)$