Recall from last time:

$$A^{\bullet}(M) \supset \bar{A}^{\bullet}(M)$$

The absence of "$\bullet$" means - $\bar{A}(M) = \bar{A}^{top}(M)$
- $A(M) = A^{top}(M)$

Pick parameters a_e, $e \in E$ and define

$$A^1(M) \ni w = \sum_{e \in E} a_e \cdot e$$

Also, $a_F := \sum_{e \in F} a_e$. For a maximal flag $F_0 \subset F_1 \subset \dots \subset F_r$
"$\emptyset$" "E"

Define: $a F_{\bullet} = \prod_{i=1}^{r-1} a_{F_i}$

Defn: (de Rham intersection form)

$$\langle \cdot, \cdot \rangle^{dR} : \bar{A}(M) \times \bar{A}(M) \longrightarrow \mathbb{C}(a_e)$$

$$\langle x, y \rangle^{dR} = \sum_{F_{\bullet} \in Fl(M)} Res_{F_{\bullet}}(x) \frac{1}{a_{F_{\bullet}}} Res_{F_{\bullet}}(y)$$

maximal flags of lattice of flats

If H is a hyperplane, $Res_H : \bar{A}^{\bullet}(M) \longrightarrow \bar{A}^{\bullet-1}(M/H)$

- $e_1 \wedge \dots \wedge e_{k-1} \wedge e \longmapsto e_1 \wedge \dots \wedge e_{k-1}$, if e corresponds to H

$-\, e_1 \wedge \cdots \wedge e_k \longmapsto 0$, if e does not appear in e_i

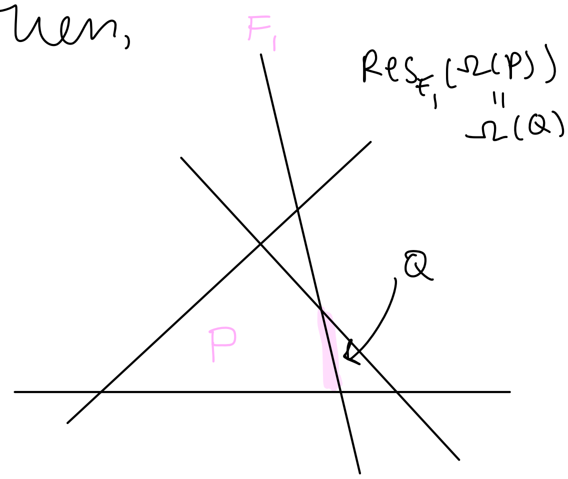
We see $\mathrm{Res}_{F_\bullet} := \cdots \circ \mathrm{Res}_{F_2} \circ \mathrm{Res}_{F_1}$

In degree 0, the Orlik-Solomon algebra "reduces" to a scalar quantity

Defn: (matroid amplitude)

Let P be one of the chambers in the hyperplane arrangement. Then,

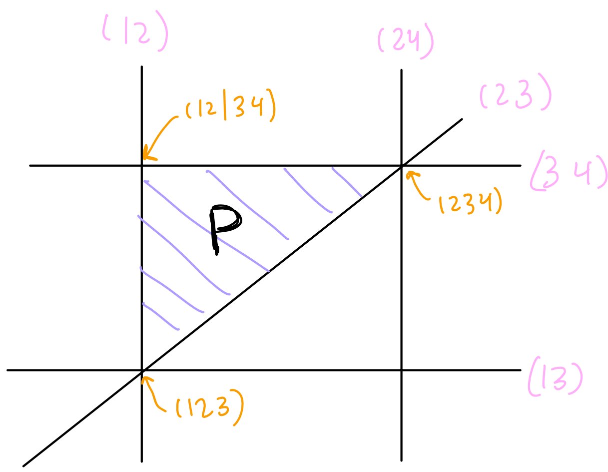
$$A(P) := \langle \Omega(P), \Omega(P) \rangle^{dR}_{\mathcal{A}} = \sum_{\substack{\text{flags}\\ \text{of flats}\\ \text{in face}\\ \text{lattice}\\ \text{of } P}} \frac{1}{a_{F_\bullet}}$$



The part of F_1 that lies on bdy. of P

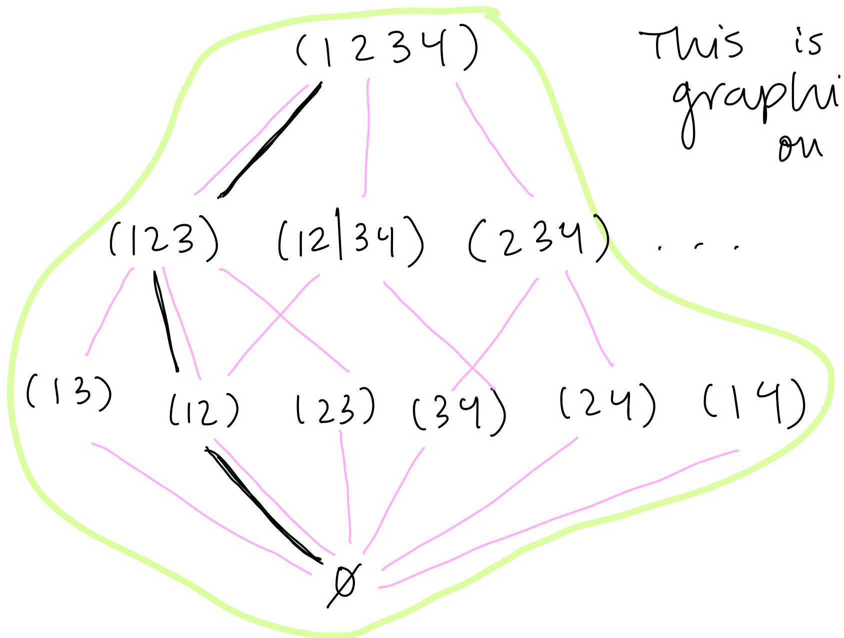
Defn: $A(P, Q) := \langle \Omega(P), \Omega(Q) \rangle^{dR}$ is the partial amplitude

Ex:



$e = (12) \Rightarrow a_{(12)} = s_{12}$

$\mathcal{L}(P)$



This is the graphic matroid on the lattice of flats (circled in green)

$$\mathcal{A}(P) = \frac{1}{s_{12}(s_{12}+s_{13}+s_{23})} + \boxed{\frac{1}{s_{12}(s_{12}+s_{34})} + \frac{1}{s_{34}(s_{12}+s_{34})}} + \frac{1}{s_{12}(s_{23}+s_{13}+s_{23})}$$

$$+ \frac{1}{s_{23}(s_{23}+s_{24}+s_{34})} + \frac{1}{s_{34}(s_{23}+s_{24}+s_{34})}$$

$$\frac{1}{s_{12}(s_{12}+s_{34})} + \frac{1}{s_{34}(s_{12}+s_{34})} = \frac{1}{s_{12}s_{34}}$$

Path in $\mathcal{L}(P)$ w/ black edges

$$= \mathcal{A}_5^{\phi_3}$$

Thm: If M is not connected, then $\mathcal{A}(P) = 0$. Otherwise, $\mathcal{A}(P)$ has simple poles only at $a_F = 0$ for connected flats F.

$$\mathrm{Res}_{a_F} \mathcal{A}(P) = \mathcal{A}(P|_F)\,\mathcal{A}(P|_{E/F})$$

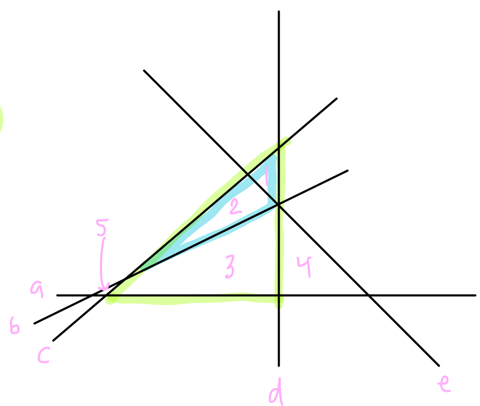
Thm: $\langle \cdot, \cdot \rangle^{dR}$ is non-degenerate on $\bar{A}(M)$ when $a_E \neq 0$.

when it is non-degenerate, it has an inverse.

Defn: $\langle P, Q \rangle_{dR} := \sum_{B \in \mathcal{B}(P,Q)} a^B \quad , \quad a^B := \prod_{b \in B} a_b$

$\hookrightarrow \mathcal{B}(P,Q) := \{ B \in \text{bases of } M \mid P, Q \text{ belong to the simplex cut out by } B \}$

Ex:

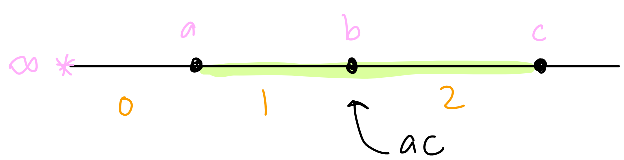


generic extension is at ∞

$$\langle \cdot, \cdot \rangle_{dR} = \begin{bmatrix} acd + bcd + cde & acd + bcd & acd & 0 & 0 \\ acd + bcd & bcd + bce + acd + ace & \cdots & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \end{bmatrix}$$

Thm: $\frac{1}{a_E} \langle \cdot, \cdot \rangle_{dR}$ is inverse to $\langle \cdot, \cdot \rangle^{dR}$ w.r.t. basis of canonical forms.

Ex:



$$\langle \cdot , \cdot \rangle^{dR} = \begin{bmatrix} \frac{1}{a}+\frac{1}{c} & -\frac{1}{a} & -\frac{1}{c} \\ -\frac{1}{a} & \frac{1}{a}+\frac{1}{b} & -\frac{1}{b} \\ -\frac{1}{c} & -\frac{1}{b} & \frac{1}{b}+\frac{1}{c} \end{bmatrix}$$

Basis from last time & it's nonsingular & $\det = \frac{a+b+c}{abc}$

↳ This arises from looking at the bounded chambers

Let's look at $\begin{bmatrix} \frac{1}{a}+\frac{1}{b} & -\frac{1}{b} \\ -\frac{1}{c} & \frac{1}{b}+\frac{1}{c} \end{bmatrix}^{-1} = \frac{1}{a+b+c} \begin{bmatrix} ab+ac & ac \\ ac & ac+bc \end{bmatrix}$

There is a classical de Rham thm that states

$$H^*(X, \mathbb{R}) \cong H^*_{dR}(X)$$

The "twisted version" states $H^*(\bar{u}, \mathcal{L}_{\bar{b}}) \cong H^*(\bar{u}, \nabla_{\bar{a}})$

concentrated in middle degree

$d + w\wedge$

Cho-Matsumoto found $\exists$ bilinear form

$$H^*(\bar{u}, \nabla_{-\bar{a}}) \times H^*(\bar{u}, \nabla_{\bar{a}}) \longrightarrow \mathbb{C}$$

$\cong$ $\cong$

$\bar{A}^d(M) / w\bar{A}^{d-1}(M)$

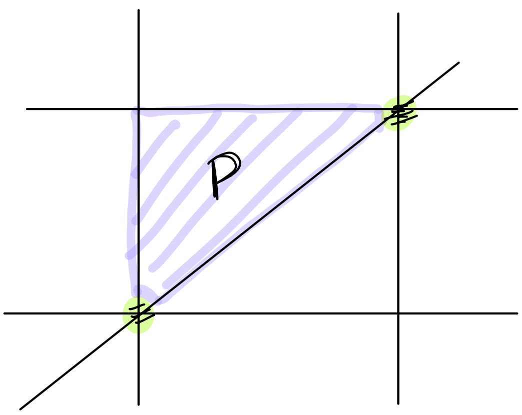
↳ The two homologies are isomorphic to the same Orlik-Solomon homology (surprising!)

Let's look at the hypergeometric angle:

$$F(a,b,c,z) = \int x^a (x-1)^b (x-z)^c \, dx$$

Now, replace by generalized hypergeometric functions

$$\int \ell_1(\underline{x})^{a_1} \ell_2(\underline{x})^{a_2} \cdots \ell_r(\underline{x})^{a_r} \, d\underline{x}$$



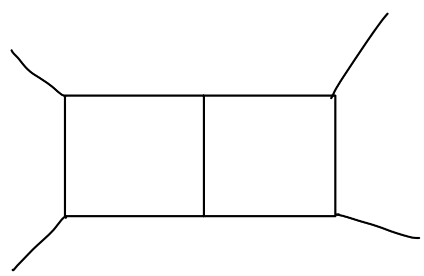
$\mathcal{M}_{0,5} \subset \overline{\mathcal{M}}_{0,5}$

The ● are places the intersections do not meet like "typical" hyperplanes
(The ● are empty)

Now, P is a pentagon. The closure of P in $\overline{\mathcal{M}}_{0,5}$ "is" an associahedron

In quantum field theory,

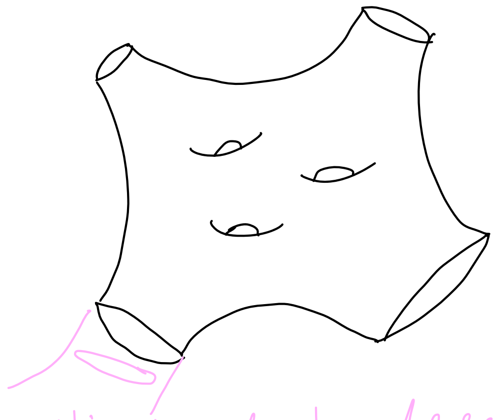
$$\mathcal{A} = \sum_{\substack{\text{loops} \\ L \geq 0}} \sum_{\substack{\text{Feynman} \\ \text{diagrams}}} (\quad)$$



In string theory,

$$\mathcal{I} = \sum_{\substack{\text{genus} \\ g \geq 0}} \int_{M_{g,n}} (\qquad)$$

scattering of strings



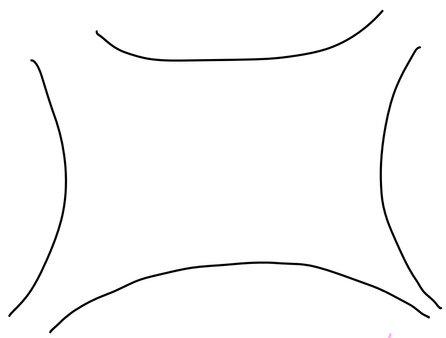
$n = 4$
$g = 3$

It's hard to describe what happens in the pink area

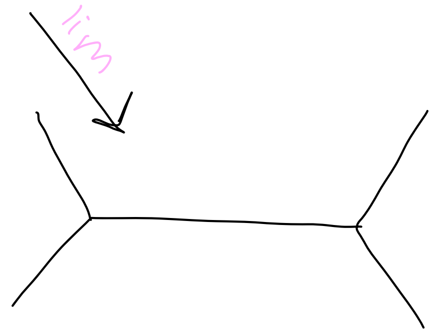
$\mathcal{I}^{closed}$ = scatter $\bigcirc$ S^1

$\mathcal{I}^{open}$ = scatter ——— $[0,1]$

↳ using open strings,



$M_{0,n}(\mathbb{R})$



"Thm": $\displaystyle\lim_{\substack{\text{string} \\ \text{length} \\ \to 0}} \mathcal{I}_n^{\text{open}, g=0} = \mathcal{A}_n^{\varphi^3, \text{tree}}$