Schubert Calculus in the Grassmannian

References:

Fulton, Young Tableaux, Chapter 9

Gillespie, Variations on a Theme of Schubert Calculus

Goal: understand linear intersection problems

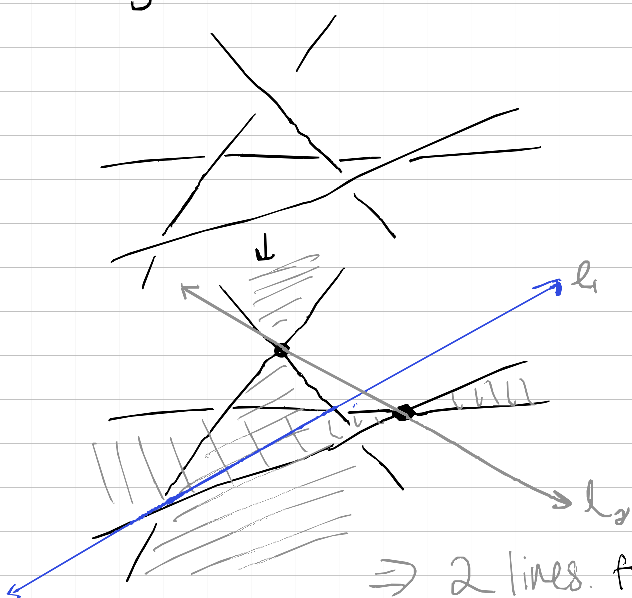
Intro Question: ① how many lines pass through 2 given points in the plane?
(1 as long as the points are distinct)

② how many points lie on 2 given lines in the plane?
(1 as long as the lines aren't parallel or equal)

note: dimension of the space of the matters here!

Harder Question: how many lines pass through 4 given lines in 3-space?
generic

sketchy solution:



⇒ 2 lines for this degenerate version

Claim: this preserves # of solutions

But how do we actually know this?

In late 1800's, early 1900's geometers were doing these types of calculations. In particular Schubert proposed this type of soln to the problem, arguing "conservation of number" + "special position"

In response to these efforts, in his 15^{th} Problem, Hilbert calls for

1) rigorous development of Schubert's calculus, +
2) a way to predict # of solns prior to doing algebraic elimination

How to formalize these problems + solve them rigorously? How do we know there is a general solution?
(any generic choice yields same intersection #)

First roadblock? How to avoid parallel issue? What if we say those parallel lines just meet at ∞?

$\boxed{\text{Def}}$ n-dim projective space (over $\mathbb{C}$) $\mathbb{P}^n$ is the set of equivalence classes in $\mathbb{C}^{n+1} - \{(0, \dots, 0)\}$ where

$$\underbrace{(x_0 : x_1 : \dots : x_n)}_{\text{point in } \mathbb{P}^n} \sim (y_0 : y_1 : \dots : y_n) \text{ if}$$

$\exists \alpha \in \mathbb{C} - 0$ s.t. $x_i = \alpha y_i \ \forall i \in [n]$

point in $\mathbb{P}^n$ $\longleftrightarrow$ line in $\mathbb{C}^{n+1}$ through origin

line in $\mathbb{R}^3$ $\to$ line in $\mathbb{P}^3$ $\to$ plane in $\mathbb{C}^4$

We'll view these problems in projective space

lines in $\mathbb{P}^3$ $\longleftrightarrow$ 2-planes in $\mathbb{C}^4$

Def The Grassmannian $Gr(k,n)$ is the set of all k-dim subspaces in $\mathbb{C}^n$.

We can associate any point in $Gr(k,n)$ with the span of k LI vectors in $\mathbb{C}^n$

represent point in $Gr(k,n)$ with a $k \times n$ matrix whose rows are these vectors

Exercise: each point in $G(k,n)$ is the rows span of a unique full-rank $k \times n$ matrix in RREF

we know each matrix will have a well-defined RREF. Are these families of matrices helpful to study?

→ form equivalence classes & call them Schubert cells

Ex)

$$\begin{bmatrix} -8 & 12 & 4 & 0 & 0 & 0 & 0 \\ 1 & 8 & 2 & 10 & -3 & 3 & 0 \\ 2 & 4 & 1 & 3 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -2 & 3 & 1 & 0 & 0 & 0 & 0 \\ -5 & -4 & -1 & 1 & 0 & 0 & 0 \\ 7 & 4 & 1 & 3 & -1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & * & \boxed{1} & 0 & 0 & 0 & 0 \\ * & * & 0 & \boxed{1} & 0 & 0 & 0 \\ * & * & 0 & 0 & * & \boxed{1} & 0 \end{bmatrix}$$

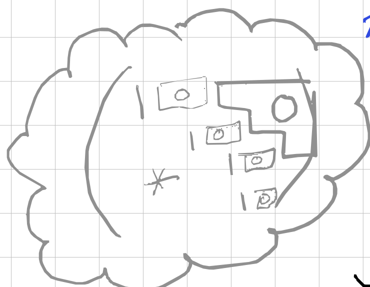
Note that $Gr(k,n)$ will be a disj union of Schubert cells

Exercise: prove it!

By structure of RREF, these look like

$$\begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & 0 & * & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 1 & 0 & 0 \\ * & 0 & * & * & 0 & * & 1 \end{bmatrix}$$

$\lambda = (2,2,1)$ $\qquad$ $\lambda = (3,1,0)$

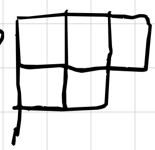
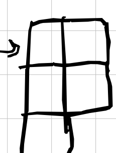


Let $\lambda_i := (\text{position of } 1 \text{ in row } i) - (k-i) - 1$

+ let Ω_λ denote the Schubert cell associated to λ.

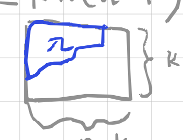
Then $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is an (integer) __partition__.

The __Young diagram__ of λ is the diagram of boxes s.t. have λ_i boxes in row i

$\lambda = (3,2,0) \rightsquigarrow$  $\qquad \lambda = (2,2,1) \rightsquigarrow$ 

Let $\underbrace{|\lambda|}_{\text{size of } \lambda} = \sum \lambda_i$ + $\underbrace{\ell(\lambda)}_{\text{length of } \lambda} = k$

Then $Gr(k,n) = \dot{\bigcup}_{\substack{\lambda_1 \leq n-k \\ \ell(\lambda) = k}} \Omega_\lambda$ $\quad$ ← we say $\lambda \leq k \times (n-k)$



But we can do this wrt any basis of $\mathbb{C}^n$

To any __ordered__ basis, we can associate a flag: $V_\bullet : \{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{C}^n$

$\{v_1, \ldots, v_n\}$

s.t. $\dim V_i = i$

+ $\langle v_1, \ldots, v_i \rangle = V_i$.

Then

$$\Omega_\lambda(E_\bullet) = \{ V \in Gr(k,n) \mid \dim(V \cap \overbrace{\langle e_1, \ldots, e_r \rangle}^{E_r}) = i \text{ for } n-k+i-\lambda_i \leq r \leq n-k+i-\lambda_{i+1} \}$$

$$\Rightarrow \dim(\Omega_\lambda) = k(n-k) - |\lambda|$$

Note: $Gr(k,n)$ is $k(n-k)$-dimensional.

The Schubert variety X_λ is

$$X_\lambda(E_\bullet) = \underbrace{\overline{\Omega_\lambda(E_\bullet)}}_{\text{zariski closure}} = \{V \in Gr(k,n) \mid \dim(V \cap \langle e_1, \dots, e_r\rangle) \geq i \text{ for } n-k+i-\lambda_i \leq r \leq n-k+i-\lambda_{i+1}\}$$

Then we see $\underset{\substack{\| \\ \{V \in Gr(2,4) \,\mid\, \dim(V \cap \langle e_1, e_2\rangle) \geq 1\}}}{\overset{X_\square(E_\bullet) \,/\!/}{X_{(1,0)}}} \subset Gr(2,4) \rightarrow$ $i=1$: $2+1-1 \leq r \leq 2+1-0 \Downarrow$

i.e. after translation the lines in $\mathbb{P}^3$ that nontrivially intersect a given line (in at least a point)

$\therefore$ our central motivating problem is looking for

$\#$ points in $\left(X_\square(F_\bullet^1) \cap X_\square(F_\bullet^2) \cap X_\square(F_\bullet^3) \cap X_\square(F_\bullet^4)\right)$

which flags to use?

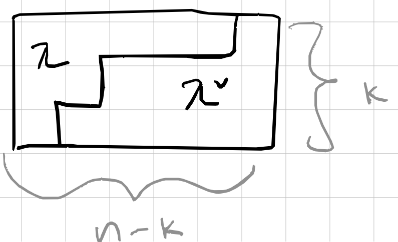
This will be well-defined for <u>transverse flags</u>, i.e. flags $F_\bullet, G_\bullet$ s.t. $\forall i,j$ if and only if $\forall F_i, G_j$:

$$\dim(F_i \cap G_j) = \max(0, \dim(F_i) + \dim(G_j) - n).$$

<u>Ex)</u> What is one such transverse pair?
$E_\bullet$ the standard flag $(E_i = \langle e_1, \dots, e_i\rangle)$
$+$ $F_\bullet$ the opposite flag $(F_i = \langle e_n, \dots, e_{n-i+1}\rangle)$

For $\lambda \subseteq k \times (n-k)$ let its complement $\lambda^\vee$ be
$\lambda^\vee = (k \times (n-k)) - \lambda$, i.e. $\lambda_i + \lambda^\vee_{k+1-i} = n-k \quad \forall i$

<u>ex)</u>



$\boxed{\text{Thm}}$ Let $F_\bullet, E_\bullet$ be transverse flags & $\lambda, \mu \subseteq k \times (n-k)$.

Then if $(X_\lambda(F_\bullet) \cap X_\mu(E_\bullet)) \neq \emptyset$, then

Further if $|\lambda| + |\mu| = k(n-k)$
& $\lambda_i + \mu_{k+1-i} \leq n-k \;\; \forall i \in [k]$,
then

$$\#\left(X_\lambda(F_\bullet) \cap X_\mu(E_\bullet)\right) = \begin{cases} 1 & \text{if } \mu = \lambda^\vee \\ 0 & \text{else} \end{cases}$$

But how to solve for these intersection #'s?

intersection theory $\rightarrow$ $\boxed{\text{cohomology } H^*(Gr(k,n))}$ $\rightarrow$ ring theory

Schubert variety X_λ $\nearrow$ $\searrow$ Schubert classes σ_λ

$H^*(Gr(k,n))$ has $\mathbb{Z}$-basis given by $\{\sigma_\lambda\}_{\lambda \subseteq k \times (n-k)}$

$$\Rightarrow \sigma_\lambda \sigma_\mu = \sum_{\substack{\nu \\ \cap \\ k\times(n-k)}} c_{\lambda\mu}^{\nu} \sigma_\nu, \quad c_{\lambda\mu}^{\nu} \in \mathbb{Z}$$

Further, $H^*(Gr(k,n))$ is commutative graded ring
$\displaystyle \bigoplus_{i=1}^{k(n-k)} H^{2i}(Gr(k,n))$ where $H^{2i}(Gr(k,n))$ is generated by classes σ_λ s.t. $|\lambda| = i$ & $\lambda \subseteq k \times (n-k)$

(⊗) Further,
$$\sigma_\lambda \cdot \sigma_\mu = [X_\lambda(F_\bullet^1) \cap X_\mu(F_\bullet^2)]$$
for transverse $F_\bullet^1, F_\bullet^2$

∴ cohomology gives rigorous foundations for int. thy computations

Then if $|\lambda_1| + |\lambda_2| + \dots + |\lambda_\ell| = k(n-k)$

$\Rightarrow \sigma_{\lambda_1} \cdot \ldots \cdot \sigma_{\lambda_\ell} \in H^{k(n-k)}(Gr(n,k))$

$\Rightarrow \sigma_{\lambda_1} \cdot \ldots \cdot \sigma_{\lambda_\ell} = c_{\lambda_1 \dots \lambda_\ell}\, \sigma_{k \times (n-k)}$ since graded ring

By (⊗) it follows
$c_{\lambda_1 \dots \lambda_\ell} = \#$points in $X_{\lambda_1}(F_\bullet^1) \cap \ldots \cap X_{\lambda_\ell}(F_\bullet^\ell)$

Using this formalization,

if $c = \#$points in $(X_\square(F_\bullet^1) \cap X_\square(F_\bullet^2) \cap X_\square(F_\bullet^3) \cap X_\square(F_\bullet^4))$

$\Leftrightarrow \sigma_\square^4 = c \cdot \sigma_{⊞}$

Q: how can we use H^* to compute c?

We know $H^*(Gr(k,n))$ is isomorphic to some ring R/I.

$$\varphi : H^*(Gr(k,n)) \xrightarrow{\cong} R/I$$
$$\sigma_\lambda \longmapsto ?$$

Q: is such R/I easy to understand? s.t. we have $\varphi(\sigma_\lambda)$ easily computable? What about their products?