Connections to Symmetric Functions

References:
Fulton, Young Tableaux, Chapters 1, 5, 6, 9.

Let $R = \mathbb{Z}[[x_1, x_2, \ldots]]$ denote the ring of formal power series.

Def We say a function $f \in R$ is <u>symmetric</u> if
$$f(x_1, x_2, \ldots\ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots)$$
$\forall$ permutations $\pi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ (permuting finite # entries)
& say $\deg(f) < \infty$.

Let Λ denote the ring of these sym fns.

Ex) $f = x_1 + x_2 x_3$ not symmetri

$$g = \sum_{i=1}^{\infty} x_i + \sum_{1 \le i < j < \infty} x_i x_j \text{ is symmetric}$$

Let $P_n = \{\lambda \text{ partition} \mid \ell(\lambda) \le n\}$
One nice $\mathbb{Z}$-basis in Λ are the <u>monomial symmetric functions</u>.

$$m_\lambda = \sum_{(i_1, \ldots, i_\ell) \in \mathbb{Z}^\ell_{\ge 0}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_\ell}^{\lambda_\ell}$$

So if $f \in \Lambda$
$f = \sum_\lambda c_\lambda m_\lambda$, $c_\lambda \in \mathbb{Z}$

By construction, m_λ are symmetric.

What about other families?

Could take complete symmetric fn.
$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell} \text{ where}$$
$$h_k = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

Ex) $h_3 = (x_1^3 + x_2^3 + \ldots) + (x_1^2 x_2 + x_1 x_2^2 + \ldots) + (x_1 x_2 x_3 + x_1 x_2 x_4 + \ldots)$
$= m_3 + m_{21} + m_{111}$

A **tableau** of shape λ is a map $T : \lambda \mapsto \mathbb{Z}_{\geq 0}$ i.e. a filling of the Young diagram of λ w/ non-neg integers

A tableau is **semistandard** if entries weakly inc. across rows & strictly inc. down cols. Let $SSYT(\lambda) = \{T \mid T$ semistandard Young tab. of shape $\lambda\}$

Ex)

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 1 & 3 \\ \hline 1 & 4 & 6 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array} \quad \text{tableau of shape } (4,3,1) \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 7 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \quad \text{SSYT}$$

If T has μ_i i's for $i \in [\ell]$, say T has **weight** $\mu = wt(T)$ (say $x^{\mu} = x_1^{\mu_1} x_2^{\mu_2} \cdots$)

Let $SSYT(\lambda, \mu) = \{T \in SSYT(\lambda) \mid T$ has weight $\mu\}$

Then define the **Schur function**

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^{wt(T)} = \sum_{\mu} \underbrace{\# SSYT(\lambda,\mu)}_{K_{\lambda\mu}} m_\mu$$

Kostka coeff

By 2nd equality, s_λ is symmetric.

Ex) $\lambda = (2,1)$ $\quad s_\lambda = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1x_2x_3 + x_1 x_3^2 + \cdots$

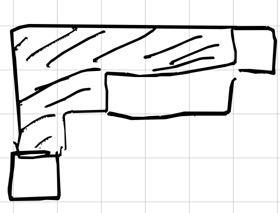
$$\begin{array}{cc}1&1\\2\end{array} \quad \begin{array}{cc}1&1\\3\end{array} \quad \begin{array}{cc}1&2\\2\end{array} \quad \begin{array}{cc}1&2\\3\end{array} \quad \begin{array}{cc}1&3\\2\end{array} \quad \begin{array}{cc}1&3\\3\end{array} \quad \begin{array}{cc}1&2\\4\end{array} \quad \begin{array}{cc}2&3\\4\end{array} \cdots$$

Additionally $\{s_\lambda\}$ form a $\mathbb{Z}$-basis for Λ as well

The **skew shape** ν/λ where $\lambda \subseteq \nu$ is the complement of λ in ν. We say ν/λ is a horiz. strip if no 2 boxes lie in same col.

Ex)

not horiz strip



horiz strip

But how does this connect to Lecture 1?

$$H^*(Gr(k,n)) \cong \Lambda \Big/ \langle s_\lambda \mid \lambda \not\subseteq k\times(n-k)\rangle$$

Schur fns indexed by λ that doesn't fit in rectangle

$$\sigma_\lambda \xmapsto{\varphi} s_\lambda$$

Thus we can try to understand RHS instead

Suppose $\sigma_\lambda \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$ & $s_\lambda s_\mu = \sum_\nu d_{\lambda\mu}^\nu s_\nu$.

<u>Claim</u>: STS $c_{\lambda(a)}^\nu = d_{\lambda(a)}^\nu$ for $a \in \mathbb{Z}_{>0}$.

Note: both rings are commutative & associative

$\boxed{\text{Thm}}$ (Comb Pieri rule) $d_{\lambda(a)}^\nu = \begin{cases} 1 & \text{if } \nu/\lambda \text{ is horiz strip} \\ 0 & \text{else} \end{cases}$

In fact, this follows from

<u>Claim</u>: $s_\lambda \cdot h_\mu = \sum_\nu K_{\nu/\lambda,\mu} s_\nu$ ← exercise

Pf: $h_{(a)} = s_{(a)}$. Then $K_{\nu/\lambda,(a)} = \begin{cases} 1 & \text{if } \nu/\lambda \text{ horiz strip} \\ 0 & \text{else} \end{cases}$

$\boxed{\text{Thm}}$ (Geom Pieri rule) $d_{\lambda(a)}^\nu = c_{\lambda(a)}^\nu$, i.e.

$$c_{\lambda(a)}^\nu = \begin{cases} 1 & \text{if } \nu/\lambda \text{ is horiz strip} \\ 0 & \text{else} \end{cases}$$

<u>Pf</u>

If $E_\bullet, F_\bullet, G_\bullet$ generic ($E_\bullet$ std, $F_\bullet$ opp, $G_\bullet$ generic), then
$X_\lambda(E_\bullet) \cap X_{(a)}(F_\bullet) \cap X_{\nu^\vee}(G_\bullet) = \begin{cases} 1 & \text{if } \nu^\vee/\lambda \text{ horiz strip} \\ 0 & \text{else} \end{cases}$

<u>Idea</u>: Similar to Duality argument

Then $\{S_{(a)}\}$ + $\{\sigma_{(a)}\}$ generate their rings

<u>Note</u>: By Giambelli's formula, we see exactly S_λ in terms of $S_{(a)}$. $\left(S_\lambda = \det\left(S_{(\lambda_i - i + j)}\right)_{i,j\in[k]}\right)$

$\Rightarrow$ $\{S_{(a)}\}$ generate $\{S_\lambda\}$

Since both LHS + RHS are assoc, comm rings + φ is a ring homomorphism this shows φ is a ring isomorphism

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Already we can now compute

$$c := \#\left(X_\square(F_\bullet^1) \cap X_\square(F_\bullet^2) \cap X_\square(F_\bullet^3) \cap X_\square(F_\bullet^4)\right) \text{ in } \underline{Gr(2,4)}$$

$$\Rightarrow \sigma_\square^4 = c\,\sigma_{\ydiagram{2,2}}$$

$$(\sigma_\square \cdot \sigma_\square)\cdot \sigma_\square \cdot \sigma_\square = \left(\left(\sigma_{\ydiagram{2}} + \sigma_{\ydiagram{1,1}}\right)\sigma_\square\right)\cdot \sigma_\square$$

$$= \left(\sigma_{\ydiagram{2,1}} + \cancel{\sigma_{\ydiagram{3}}} + \cancel{\sigma_{\ydiagram{1,1,1}}} + \sigma_{\ydiagram{2,1}}\right)\sigma_\square$$

$$= 2\sigma_{\ydiagram{2,1}} \cdot \sigma_\square = 2\sigma_{\ydiagram{2,2}}$$

$\Rightarrow$ the answer must be $c = 2$

But what if we want to compute

$\sigma_\lambda \cdot \sigma_\mu$ where λ, μ are not just rows?

Since $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$

want some way to 'multiply' tableaux & expand into tableaux.

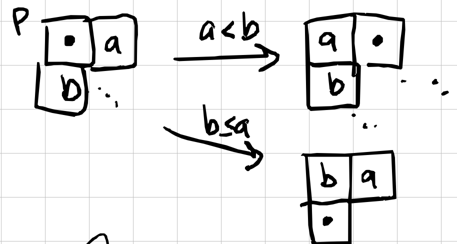
We want to map

$$\Psi : SSYT(\lambda) \times SSYT(\mu) \longrightarrow Q \in SSYT(\nu)$$

s.t. $|\Psi^{-1}(Q)| = c_{\lambda\mu}^{\nu}$ (where commutat is apparent)

Say $T \in SSYT(\lambda)$, $U \in SSYT(\mu)$.
We define $T * U$ via an iterative algorithm $Rect(V)$ where V is a skew semistandard YT.

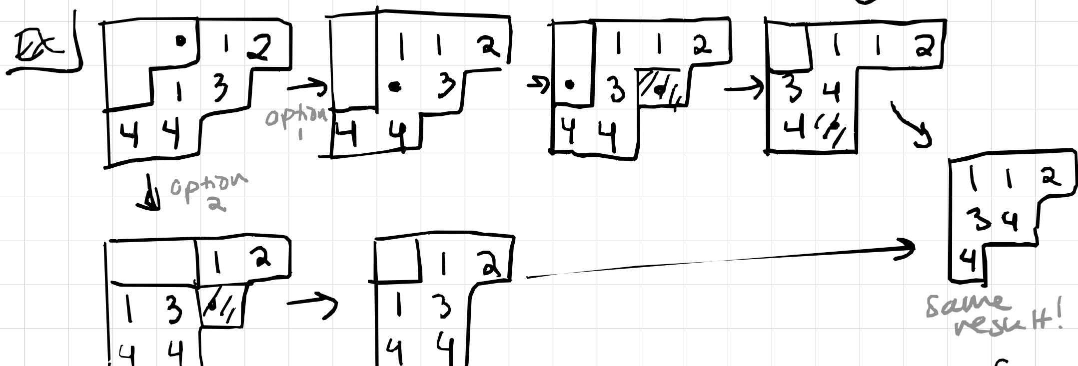
Suppose we have an inner corner of p:



continue until no boxes SW of •

Then repeat with another inner corner until none exist, i.e. the tableaux is no longer skew

Ex



Challenge: prove order of rectification doesn't matter

$\boxed{\text{Thm}}$ $c^{\nu}_{\lambda\mu} = \#\{(T,U) \in SSYT(\lambda) \times SSYT(\mu) \mid Rect(T*U) = Q\}$ for $Q \in SSYT(\nu)$ fixed

In fact, there are several combinatorial rules that allow us to compute $c^{\nu}_{\lambda\mu}$.

One way to get a clever short pf via Stembridge:

$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

Let $a_\lambda = \det\left(x_i^{\lambda_j}\right) = \sum_{w \in S_n} \varepsilon(w)\, x^{w \cdot \lambda}$

$\rho = (n-1, n-2, \dots, 2, 1)$

$\boxed{\text{Thm}}$ For all partitions $\lambda \in P_n$, partition μ

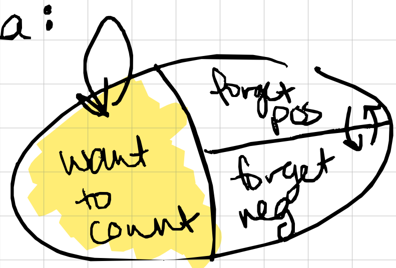
$$a_{\lambda+\rho}\, s_\mu = \sum_{\substack{T \in SSYT(\mu), \\ \lambda + wt(T_{\geq j}) \in P_n,\ \forall j}} a_{\lambda + wt(T) + \rho}$$

T restr to entries $\geq j$

In the exercises, we'll go step-by-step through a proof of this fact

The proof we'll use is by "sign-reversing involution"

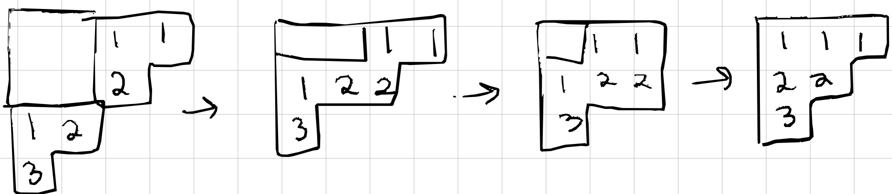
idea:

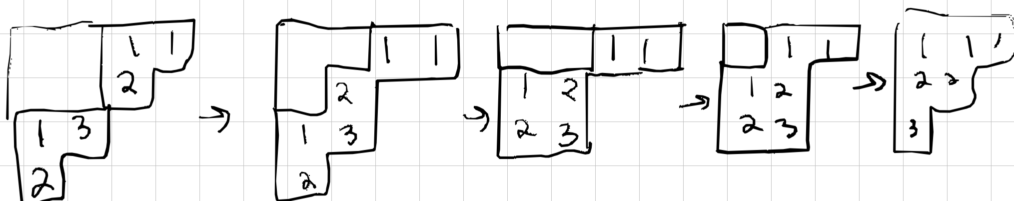


Next time: other places we see $c^{\nu}_{\lambda\mu}$

Ex What is $c^{(3,2,1)}_{(2,1),(2,1)}$?

Want $\#\left\{T \in SSYT(\lambda * \mu) \;\middle|\; Rect(T) = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \cline{1-2} 3 & & \\ \cline{1-1}\end{array}\right\}$





Only possible options

$\Rightarrow$ answer is 2