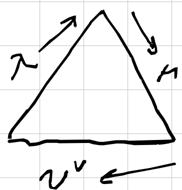
Puzzles, Saturation, & Vanishing

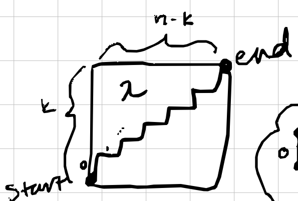
References:

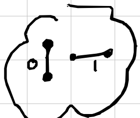
Bhatia, Linear Algebra to Quantum Cohomology: The Story of Alfred Horn's Inequalities

Different rules for LR-coeffs exhibit different geometric properties

Another ex: Knutson-Tao ('00) introduce a new rule in terms of puzzles, i.e. tilings of an $n \times n \times n$ triangular grid labelled by binary strings representing λ, μ, ν.







The tiling uses pieces

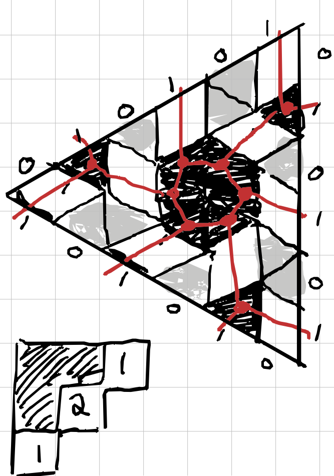
+ their rotations

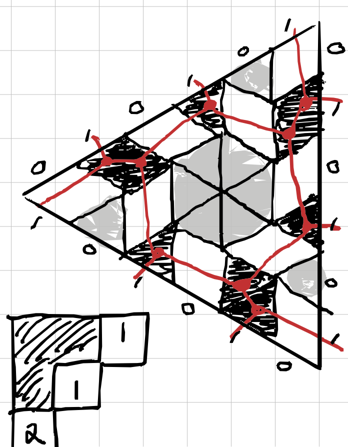
Ex: for $\lambda = (2,1)$ $\mu = (2,1)$ $\nu = (3,2,1)$ in $Gr(3,6)$

We saw

$c_{\lambda\mu}^{\nu} = 2$ using tableaux.

These correspond to puzzles





Let's talk about another fundamental problem:
the Horn eigenvalue problem

Recall: an $n \times n$ matrix A over $\mathbb{C}$ is <u>Hermitian</u> if it is self-adjoint, i.e. equal to its conjugate-transpose

ex) $\begin{pmatrix} 3 & 1-i & -7i \\ 1+i & 0 & 3i \\ -7-i & -3 & i1 \end{pmatrix}$ is Hermitian

$\boxed{\text{Problem}}$ Consider Hermitian matrices A, B, C s.t. $A+B=C$, where α, β, γ are their respective eigenvalues. Then how must γ relate to α, β?

(α_j is j^{th} largest eig. of α)

Observation 1: we know $tr(A+B) = tr(C)$
"trace eq"

$$\sum_{i=1}^{n} \alpha_i + \beta_i = tr(A) + tr(B) \qquad \sum_{i=1}^{n} \gamma_i$$

But there must be more!

$\boxed{\text{Thm}}$ (Spectral Thm) $\exists$ unitary matrix U s.t. (i.e. $U^{-1} = U^*$)

$$UAU^* = \alpha I_n = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}, \text{ i.e.}$$

A can be diag in some ON basis.

Using this ON basis, we can prove
$$\gamma_1 \le \alpha_1 + \beta_1$$
$$\gamma_n \ge \alpha_n + \beta_n .$$

Exercise: prove it!

We can generalize these ideas into a "minmax principle" by Weyl (1912)

From this, similar as above, we can obtain

$$\gamma_{i+j-1} \le \alpha_i + \beta_j \quad , \qquad i+j-1 \le n$$

More and more ineq discovered. But which fully characterize the problem?

$\boxed{\text{Thm}}$ (Horn) (1962)

$$\sum_{k\in K} \gamma_k \le \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j \quad (\otimes)$$

where $I, J, K \subseteq [n]$ s.t. $|I| = |J| = |K| = r \in \mathbb{Z}_{\ge 0}$
s.t. $(I, J, K) \in T_r^n$ some set.

Say (I, J, K) is admissible if $(\otimes)$ holds.

Horn conjectured that the following set T_r^n below is admissible + these characterize the eigenvalue problem.

(say $I = (i_1 < i_2 < \dots < i_r)$)

T_r^n is described inductively:

For $r=1$: $(I, J, K) \in T_1^n$ if $k_1 = i_1 + j_1 + 1$.

For $r>1$: $(I, J, K) \in T_r^n$ if

$$\sum_{s=1}^{r} k_s + \binom{r+1}{2} = \sum_{s=1}^{r} i_s + \sum_{s=1}^{r} j_s$$

and $\forall\, 1 \le p \le r-1$ + $(I', J', K') \in T_p^r$,

$$\sum_{k' \in K'} k_{k'} + \binom{p+1}{2} = \sum_{i' \in I'} i_{i'} + \sum_{j' \in J'} j_{j'}$$

board

By the structure of T_r^n, if true, this shows that the eigenvalues obey a natural natural convexity property + are characterized by it.

Using minmax principle + orthogonal projections, results of Weilandt '50 + Hersch-Zwahlen '62 give that if $U_\bullet, V_\bullet, W_\bullet$ are flags determined by the ON bases, and we suppose

$$L \in X_{\tilde{I}}(U_\bullet) \cap X_{\tilde{J}}(V_\bullet) \cap X_K(W_\bullet) \neq \emptyset \quad (\circledast)$$

Then $\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \geq \sum_{k \in K} \gamma_k$. Thus, as noted by Thompson '88, Helmke-Rosenthal '95,

$(\circledast)$ holds for $(\tilde{I}, \tilde{J}, K) \Rightarrow (I, J, K)$ admissible

Work of many show that the vanishing of $c_{\lambda\mu}^{\nu}$ control the Horn problem.

Klyachko '98 produced another set of inequalities. He showed that they imply Horn's + characterize the problem IF the following holds:

uses representation theory to derive

Theorem (Knutson-Tao '99)

$\boxed{\text{~~Conjecture~~}}$ (Saturation)

$$c_{\lambda\mu}^{\nu} \neq 0 \iff \text{for } \cancel{\text{some}}\ \text{all } N > 1,\ c_{N\cdot\lambda, N\cdot\mu}^{N\cdot\nu} \neq 0$$

Ex $\nu = (3,2)$
$3\cdot\nu = (9,6)$

But we have a combinatorial rule for this! How hard can it be?

($\Rightarrow$) not too bad

($\Leftarrow$) Hard! (How to know we can uniformly "shrink" a tableau?)

What to do?

Perhaps using another POV, the answer will be more clear?

Idea: Knutson-Tao use a "dual" object of honeycombs for their puzzles which exhibit this scaling more naturally.

Why else might we care about saturation?

<u>Q</u>: How efficiently can we tell if $c_{\lambda\mu}^{\nu} > 0$?

easy to check given some evidence (give me an lattice SSYT that I can check is counted by $c_{\lambda\mu}^{\nu}$)

$\Rightarrow$ the problem is in NP

how easy to determine from scratch?

(Can we decide in polynomial time?)

$\hookrightarrow$ Yes [DeLoera-McAllister '12, Narayanan-Mulmuley-Sohoni '06]

<u>Idea</u>:

Encode set of lattice tableaux + determine if this set is empty

Let $r_{ik}(T) = \#\ k$'s in row i of $T \in SSYT(\nu/\lambda, \mu)$
$\quad \| $
$\quad r_{ik}$

Take $n = \max(\ell(\nu), \ell(\mu))$.

Ex) $T = \begin{array}{|c|c|c|c|} \hline \ & \ & 1 & 1 \\ \hline \ & 1 & 2 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \rightsquigarrow (r_{ik}) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

We'll need:

$$(\star)\begin{cases} \text{- non-negative: } & r_{ik} \geq 0 & \forall i,k \in [n] \\ \text{- shape: } & \lambda_i + \sum_k r_{ik} = \nu_i & \forall i \in [n] \\ \text{- content: } & \sum_i r_{ik} = \mu_k & \forall k \in [n] \\ \text{- semistd: } & \bullet\ r_{ik} = 0 \text{ if } i<k & \\ & \bullet\ \lambda_{i+1} + \sum_{k' \leq k} r_{i+1,k'} \leq \lambda_i + \sum_{k'<k} r_{ik'}, & i,k \in [n] \\ \text{- lattice: } & \sum_{i'<i} r_{i'k} \geq r_{i\,k+1} + \sum_{i'<i} r_{i'\,k+1} & \forall i,k \in [n] \end{cases}$$

Define $P_{\lambda\mu}^{\nu} = \{(r_{ik}) \in \mathbb{R}^{n^2} \mid r_{ik} \text{ satisfy } (\star)\ \forall i,k \in [n]\}$

<u>Claim</u>: $P_{\lambda\mu}^{\nu} \cap \mathbb{Z}^{n^2} = \emptyset \iff c_{\lambda\mu}^{\nu} = 0$

pf: Exercise

Then to show $c_{\lambda\mu}^{\nu} \neq 0$ it suffices to find an integer point in $P_{\lambda\mu}^{\nu}$

How hard could it be?

In general, this is called Integer Linear Programming which is hard?

Can we get around this?

Are there special properties of this system?

We see all coeffs + constants are integers

$\therefore$

$$P_{\lambda\mu}^{\nu} \neq \emptyset \overset{\text{Exer}}{\iff} P_{\lambda\mu}^{\nu} \text{ contains a rational vertex}$$

$$\underbrace{\{N\alpha \mid \alpha \in P_{\lambda\mu}^{\nu}\}}_{\text{dilation}}$$

exercise: $N \cdot P_{\lambda\mu}^{\nu} = P_{N\lambda, N\mu}^{N\nu}$

$$\iff N \cdot P_{\lambda\mu}^{\nu} \cap \mathbb{Z}^{n^2} \neq \emptyset \text{ for some } N \in \mathbb{Z}_{>0}$$

$$\iff P_{N\lambda, N\mu}^{N\nu} \cap \mathbb{Z}^{n^2} \neq \emptyset \text{ for some } N \in \mathbb{Z}_{>0}$$

$$\iff c_{N\lambda, N\mu}^{N\nu} \neq 0 \text{ for some } N \in \mathbb{Z}_{>0}$$

$$\overset{\text{saturation}}{\iff} c_{\lambda\mu}^{\nu} \neq 0$$

$\therefore$ We just have to check if $P_{\lambda\mu}^{\nu} \neq \emptyset$ (find if there is ANY solution to this system of eqns)

This is a Linear Programming Problem

$\Rightarrow$ can use ellipsoid method to decide if $c_{\lambda\mu}^{\nu} = 0$.