Webs and alternating sign matrices

Lecture 3: Alternating sign matrix dynamics and their web connections

CMND Summer School

Jessica Striker

North Dakota State University

July 2026

Clarifications (by popular demand) on webs and invariants

Let $V = \mathbb{C}^r$ and V_i a fundamental representation: $V^{\omega_j} = \bigwedge^j V$ or V^*.

We consider $Inv_{SL_r}(V_1 \otimes \cdots \otimes V_n) := Hom_{SL_r}(V_1 \otimes \cdots \otimes V_n, \mathbb{C})$

Let W be an SL_r-web. Then the invariant is given by:

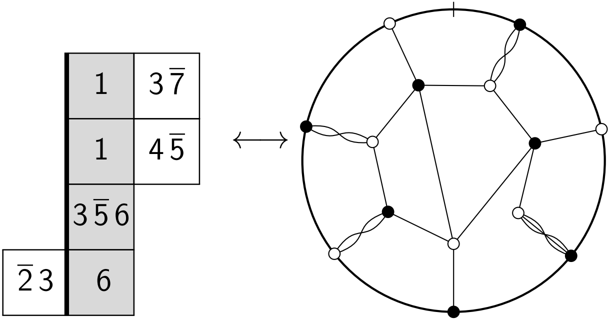
$$[W] = \sum_{\phi} (-1)^{\ell_W(\phi)} x_{\partial(\phi)}$$

where ϕ ranges over the proper labelings of W with $\{1, \ldots, r\}$, $\ell_W(\phi)$ is the *coinversion number* of ϕ (computed by summing the coinversion numbers of the permutation of edge labels read clockwise around each internal vertex), $\partial(\phi)$ is the boundary word of ϕ, and $x_w := x_{w_1,1} \cdots x_{w_n,n}$.

Rotation-invariant web bases for SL_4

Theorem (Gaetz, Pechenik, Pfannerer, S., Swanson 2023+)

- *Contracted, fully-reduced, top 4-hourglass plabic graphs with $4c$ boundary vertices give a rotation-invariant web basis for SL_4.*
- *Promotion on 4 row fluctuating tableaux $\leftrightarrow$ rotation of hourglass plabic graph equivalence classes.*
- *For $i = 1, 2, 3$, $\mathrm{trip}_i(G) = \mathrm{prom}_i(T)$.*



Proof outline (see Section 6 of [GPPSS26])

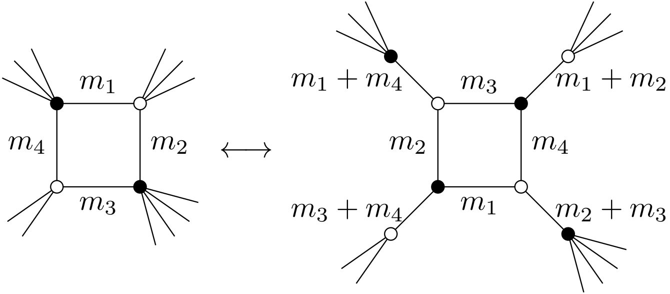
- Trip-theoretic fully reduced means move equivalent to a web with a bad face. Proof by an intricate induction on the SSV.
- The separation labeling and growth algorithm are well-defined mutually-inverse bijections:
  - The separation labeling is proper. Pf: Diagrammatic argument showing labels on either side of an hourglass are the same.
  - The separation labeling has Yamanouchi boundary word so it corresponds to a tableau and has promotion = rotation. Proof by complicated induction: Look at the endpoints of the trips starting at 1 to find the spots ("balance points") where the Yamanouchi word changes when you do promotion. Analyze subwebs along these and other contours and use induction.
  - The separation labeling of boundary edges is determined by antiexc of trip perms, while tableaux are determined by antiexcedances of their promotion perms (careful analysis of promotion in [GPPSS24]).
  - Two fully reduced HPGs of the same type are move-equivalent if and only if they have the same tuple of trip permutations. Proof: The underlying plabic graphs are move equivalent with plabic moves, so their trip_1s are equal; then do another careful analysis of the SSV.

Proof outline, continued:

- $\text{prom}_i(T) = \text{trip}_i(\mathcal{G}(T))$ comes from properties of the growth rules which come from crystals. (There's a video of Stephan Pfannerer's talk on this at the 2026 BIRS workshop on Cluster Algebras, Webs, and Canonical Bases.)
- The growth algorithm properties also show the tensor invariant of a fully reduced HPG has a leading term given by the boundary word of its separation labeling. So the invariants have distinct leading terms and are linearly independent.
- Since we have a bijection to tableaux, there are the right number of them, so our linearly independent set is a basis.
- Square moves don't change the invariant.
- We can choose a "top" benzene representative by lattice properties of benzene regions.

Web bases for SL_5? and beyond?

Guiding combinatorics: Promotion $\leftrightarrow$ rotation, Trip = Prom

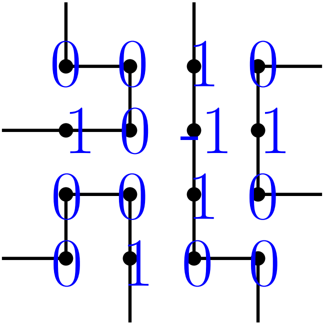


+ benzene move analogue
+ other more complicated moves

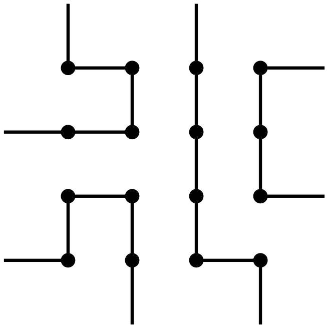
Dynamics on alternating sign matrices

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Alternating sign matrix $\leftrightarrow$ fully-packed loop

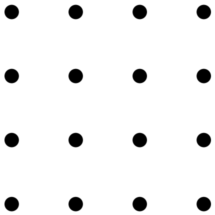


Fully-packed loop



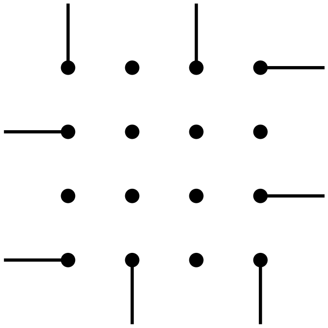
Fully-packed loops

Start with an $n \times n$ grid.



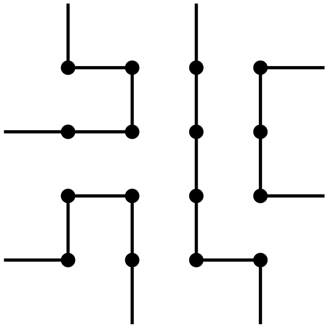
Fully-packed loops

Add boundary conditions.



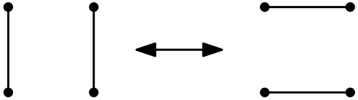
Fully-packed loops

Interior vertices adjacent to 2 edges.

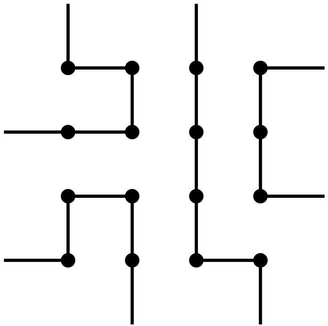


Dynamics: Gyration on fully-packed loops

The local move.

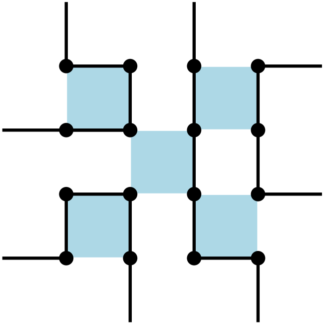


Dynamics: Gyration on fully-packed loops



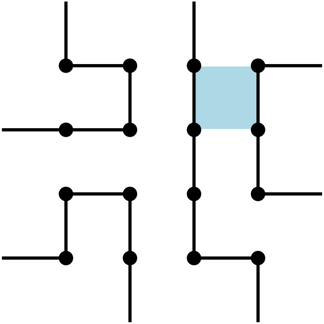
Dynamics: Gyration on fully-packed loops

Start with the even squares.



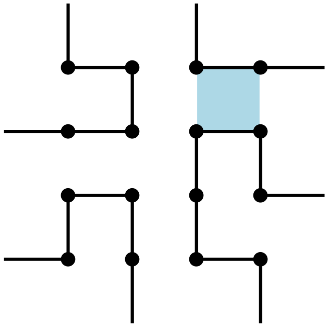
Dynamics: Gyration on fully-packed loops

Apply the local move.



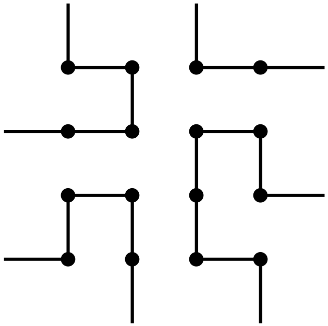
Dynamics: Gyration on fully-packed loops

Apply the local move.



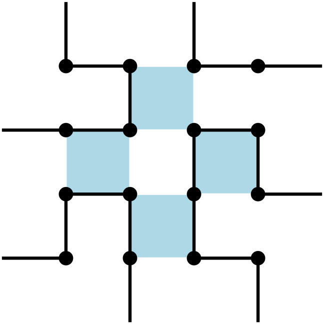
Dynamics: Gyration on fully-packed loops

Apply the local move.



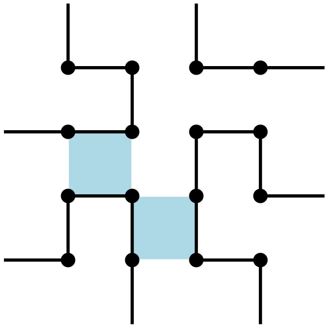
Dynamics: Gyration on fully-packed loops

Now consider the odd squares.



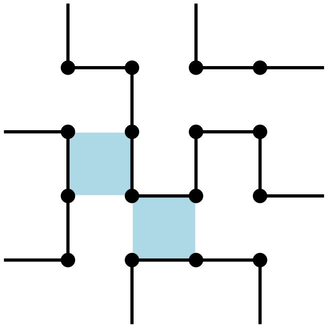
Dynamics: Gyration on fully-packed loops

Apply the local move.



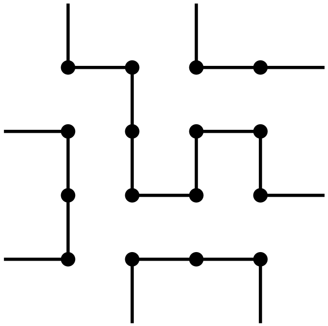
Dynamics: Gyration on fully-packed loops

Apply the local move.

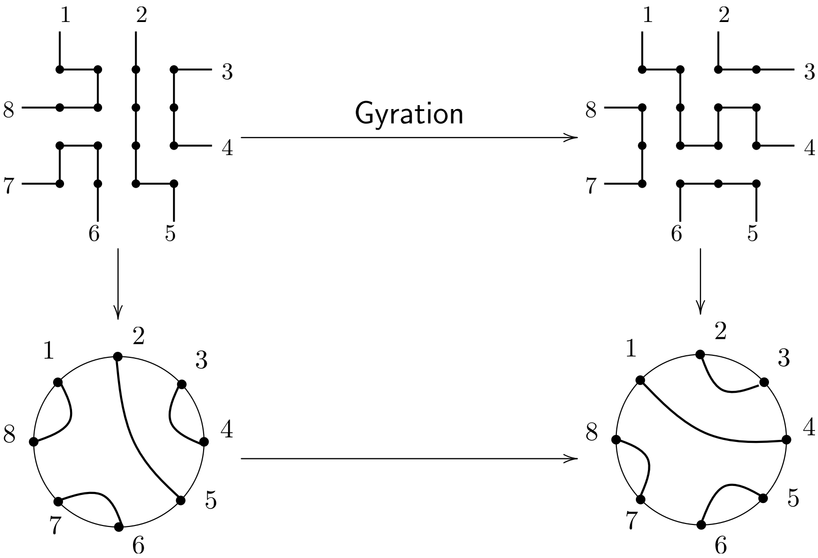


Dynamics: Gyration on fully-packed loops

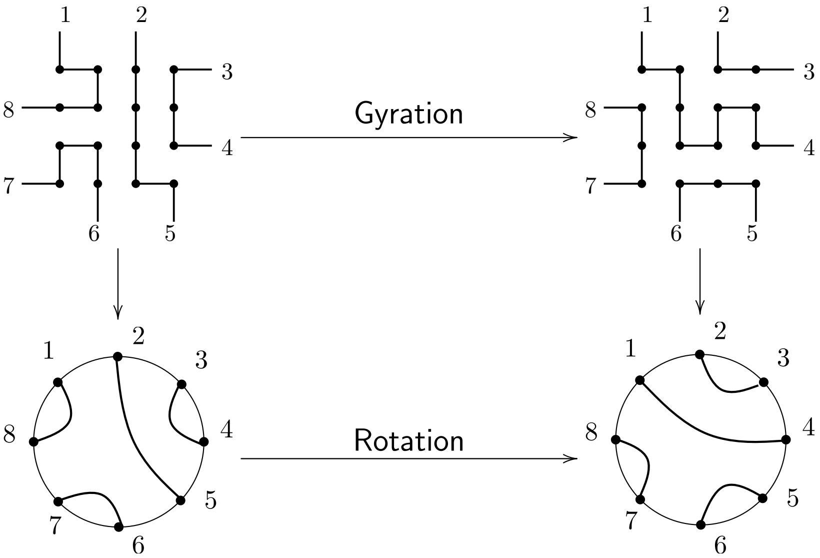
Apply the local move.



Dynamics: Gyration on fully-packed loops



Dynamics: Gyration on fully-packed loops



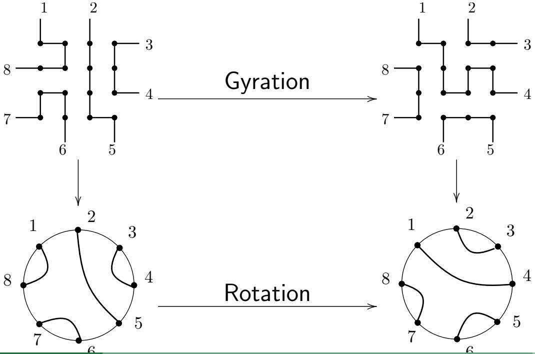
The square is a circle

Theorem (Wieland 2000)

Gyration on an $n \times n$ fully-packed loop rotates the **link pattern** *by an angle of $2\pi/2n$.*

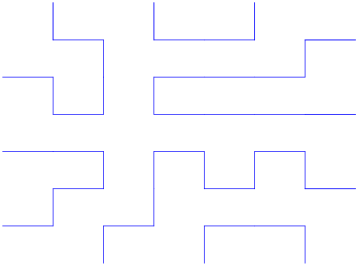
Corollary (Dilks-Pechenik-S. 2017)

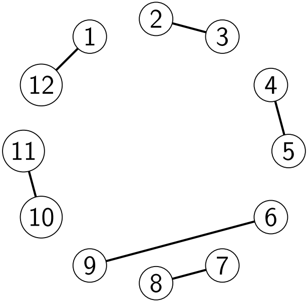
Gyration exhibits **resonance** *with frequency $2n$.*



A 6×6 ASM with gyration orbit of length 84

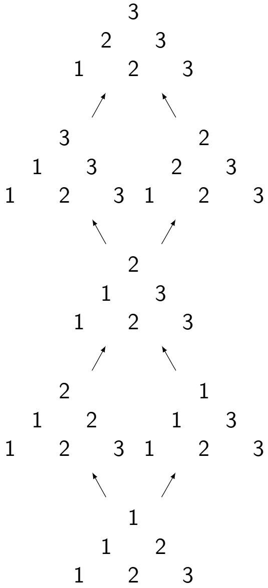
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$





Exercise: Compute gyration on a fully-packed loop; see that the link pattern rotates.

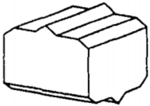
A partial order on alternating sign matrices

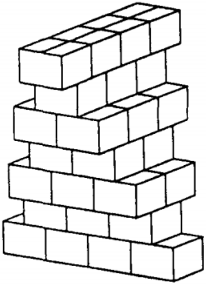


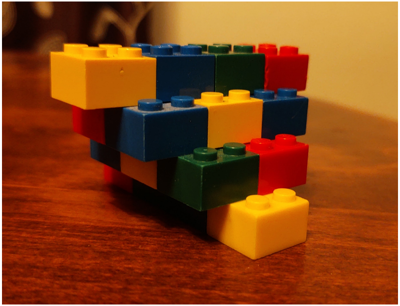
A partial order on alternating sign matrices

Theorem (Elkies, Kuperberg, Larsen, and Propp 1992)

The partial order on alternating sign matrices is a distributive lattice (that is, a lattice of order ideals) with a nice structure.

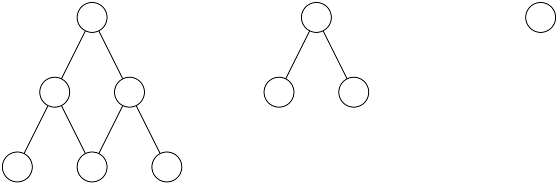




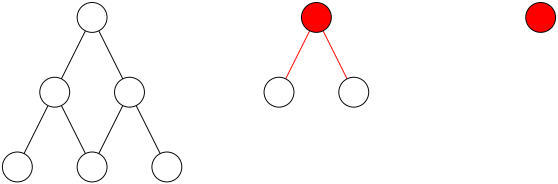


https://arxiv.org/pdf/math/9201305.pdf

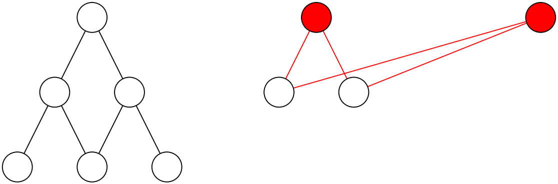
Alternating sign matrix tetrahedral poset



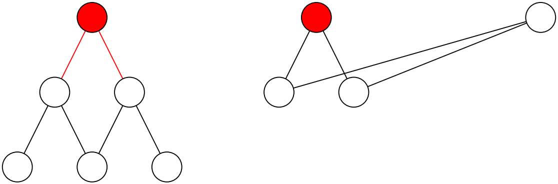
Alternating sign matrix tetrahedral poset



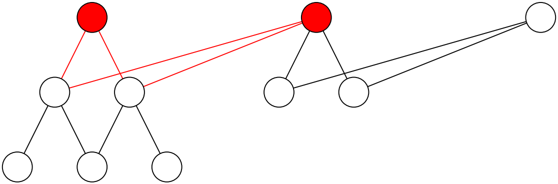
Alternating sign matrix tetrahedral poset



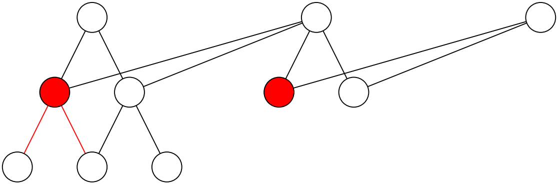
Alternating sign matrix tetrahedral poset



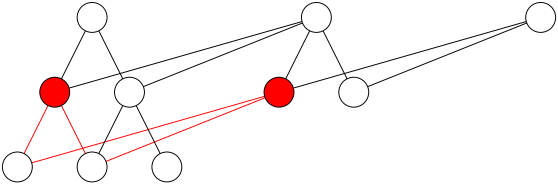
Alternating sign matrix tetrahedral poset



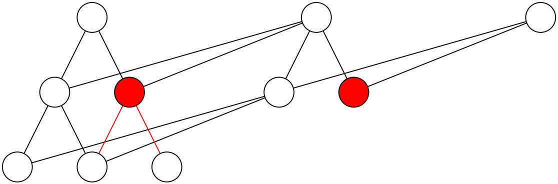
Alternating sign matrix tetrahedral poset



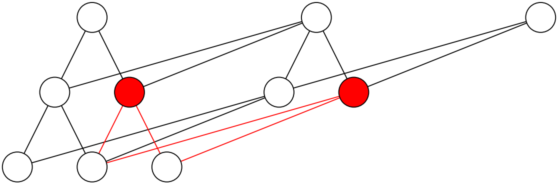
Alternating sign matrix tetrahedral poset



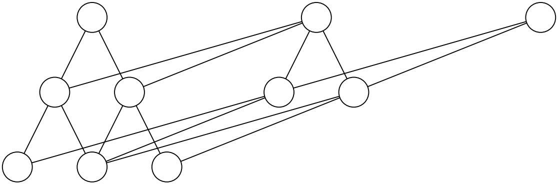
Alternating sign matrix tetrahedral poset



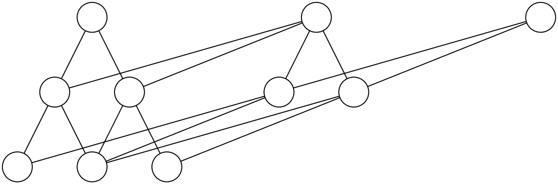
Alternating sign matrix tetrahedral poset



Alternating sign matrix tetrahedral poset



Alternating sign matrix tetrahedral poset

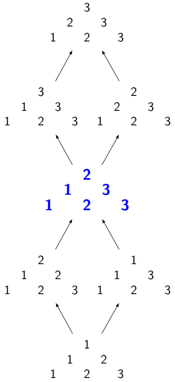


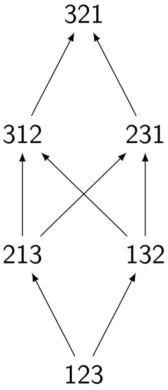
$n \times n$ alternating sign matrices are in bijection with order ideals in this $(n-1)$-layer poset.

The alternating sign matrix lattice

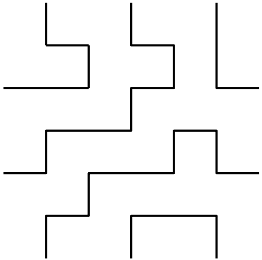
Theorem (A. Lascoux and M.-P. Schützenberger 1996)

The induced subposet of the alternating sign matrix lattice on permutations of n is the strong Bruhat order.

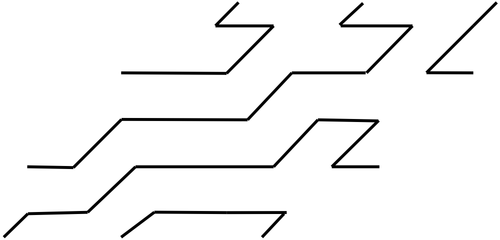




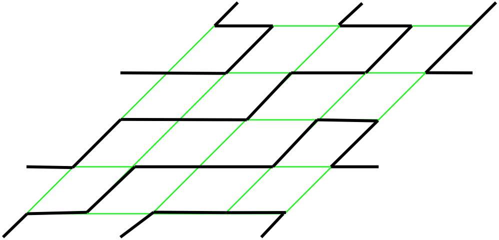
Dynamics: Gyration on the alternating sign matrix poset



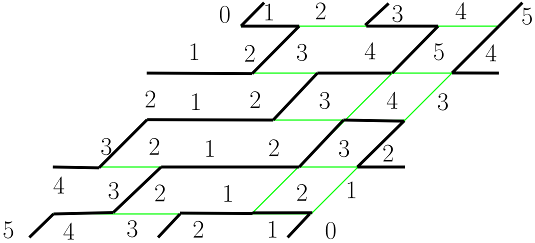
Dynamics: Gyration on the alternating sign matrix poset



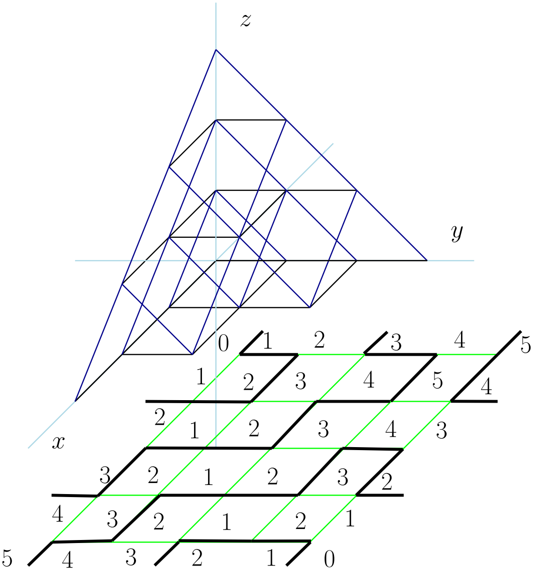
Dynamics: Gyration on the alternating sign matrix poset



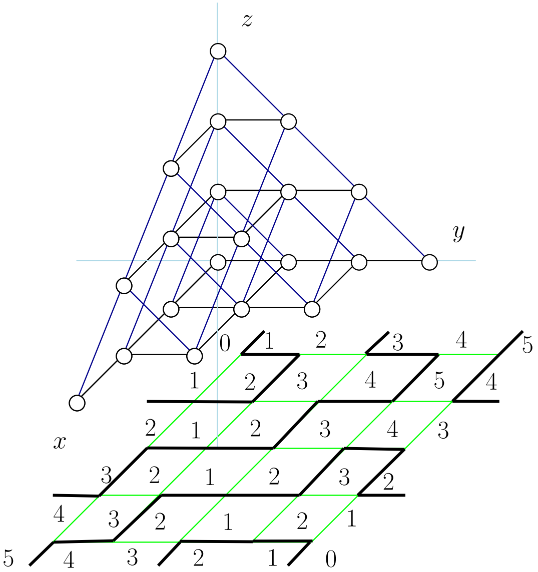
Dynamics: Gyration on the alternating sign matrix poset



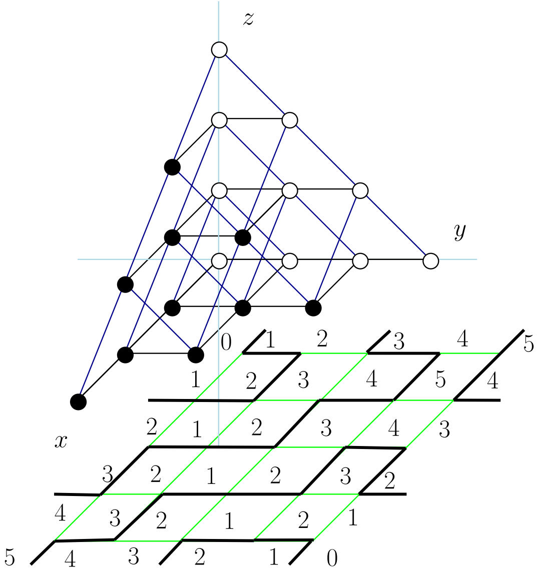
Dynamics: Gyration on the alternating sign matrix poset



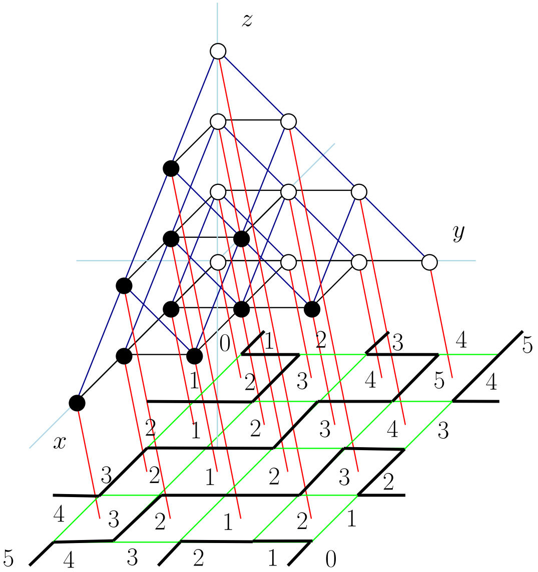
Dynamics: Gyration on the alternating sign matrix poset



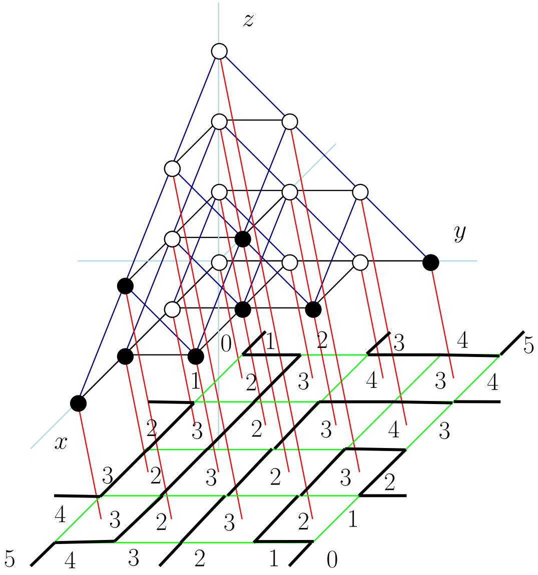
Dynamics: Gyration on the alternating sign matrix poset



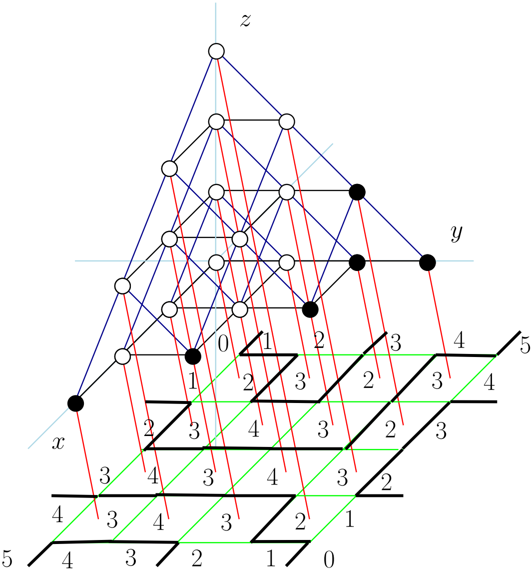
Dynamics: Gyration on the alternating sign matrix poset



Dynamics: Gyration on the alternating sign matrix poset



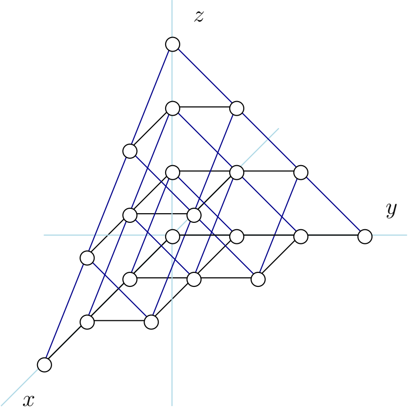
Dynamics: Gyration on the alternating sign matrix poset



Dynamics: Gyration on the alternating sign matrix poset

Theorem (N. Williams and S. 2012)

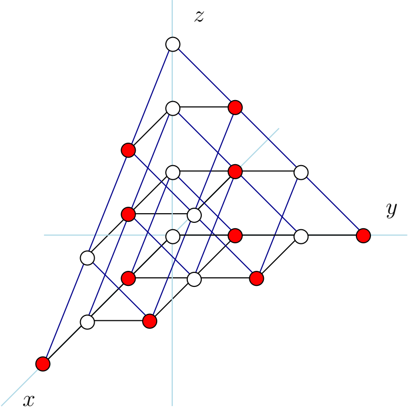
Gyration on fully-packed loops is equivalent to **toggling** *even then odd ranks in the ASM tetrahedral poset.*



Dynamics: Gyration on the alternating sign matrix poset

Theorem (N. Williams and S. 2012)

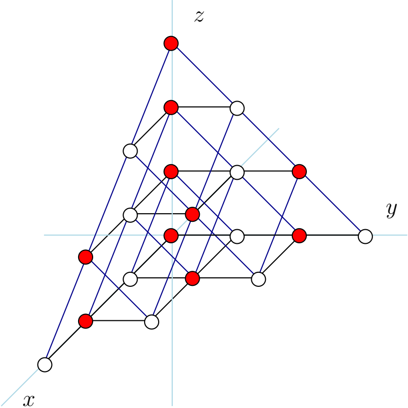
Gyration on fully-packed loops is equivalent to **toggling** *even then odd ranks in the ASM tetrahedral poset.*



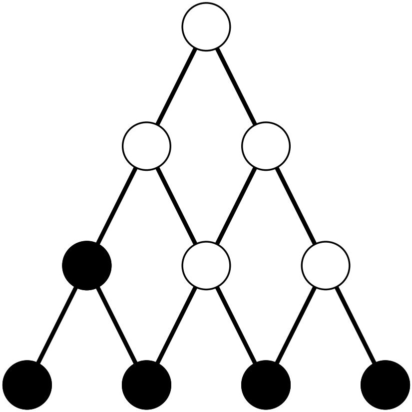
Dynamics: Gyration on the alternating sign matrix poset

Theorem (N. Williams and S. 2012)

Gyration on fully-packed loops is equivalent to **toggling** *even then odd ranks in the ASM tetrahedral poset.*

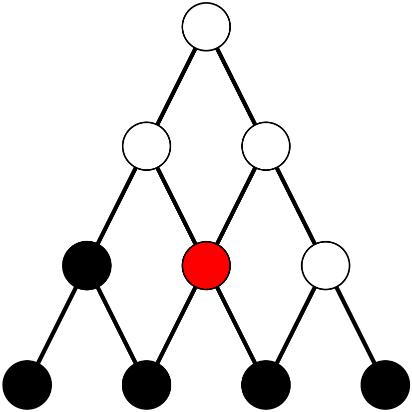


Toggles act on order ideals



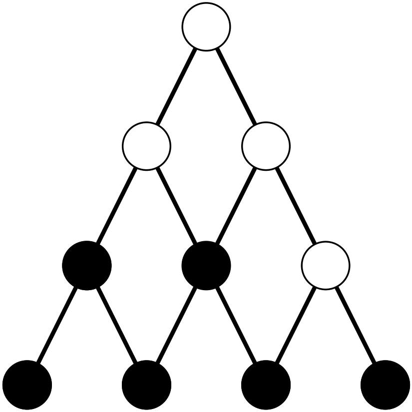
Define a toggle, t_e, for each $e \in P$.

Toggles act on order ideals



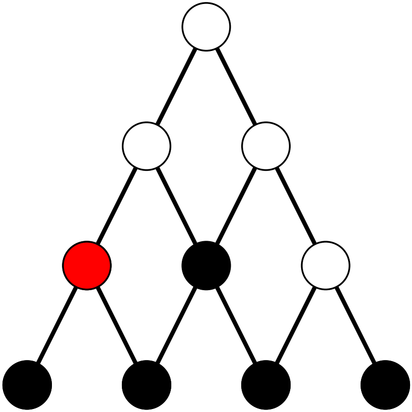
Toggles t_e add e when possible.

Toggles act on order ideals



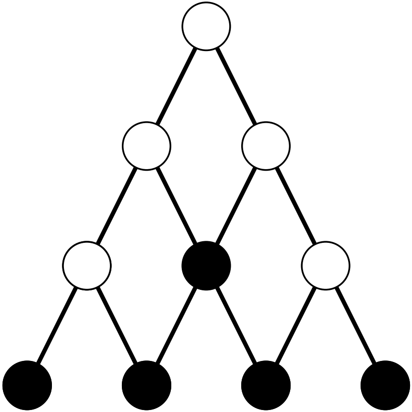
Toggles t_e add e when possible.

Toggles act on order ideals



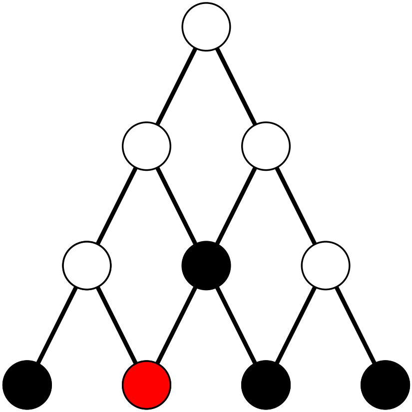
Toggles t_e remove e when possible.

Toggles act on order ideals



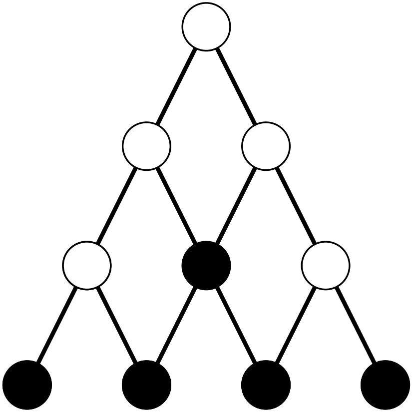
Toggles t_e remove e when possible.

Toggles act on order ideals



Toggles t_e do nothing otherwise.

Toggles act on order ideals

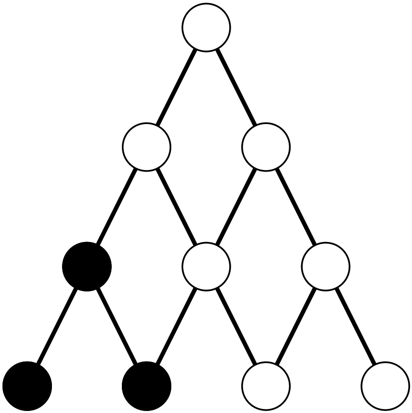


Toggles t_e do nothing otherwise.

Rowmotion

Toggle group actions are compositions of toggles.

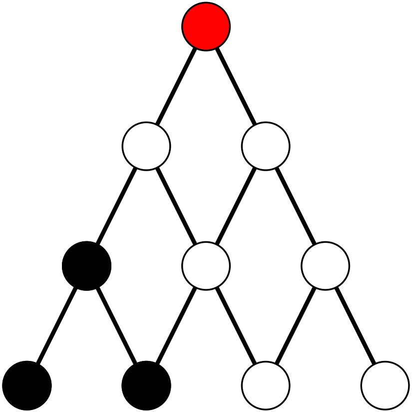
Rowmotion is the toggle group action that composes all the toggles from top to bottom.



Rowmotion

Toggle group actions are compositions of toggles.

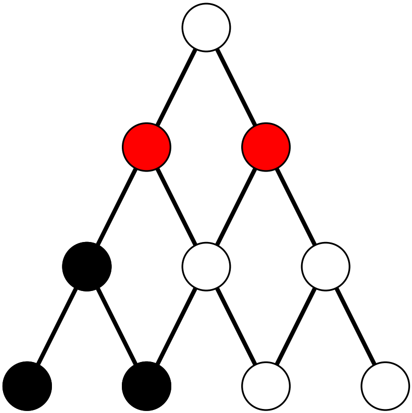
Rowmotion is the toggle group action that composes all the toggles from top to bottom.



Rowmotion

Toggle group actions are compositions of toggles.

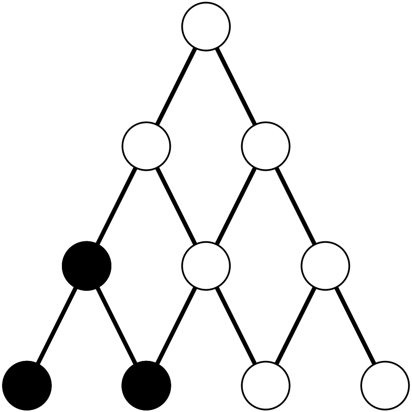
Rowmotion is the toggle group action that composes all the toggles from top to bottom.



Rowmotion

Toggle group actions are compositions of toggles.

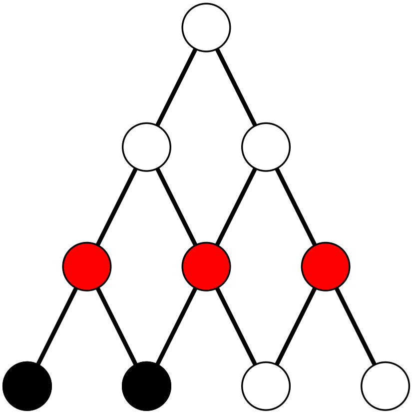
Rowmotion is the toggle group action that composes all the toggles from top to bottom.



Rowmotion

Toggle group actions are compositions of toggles.

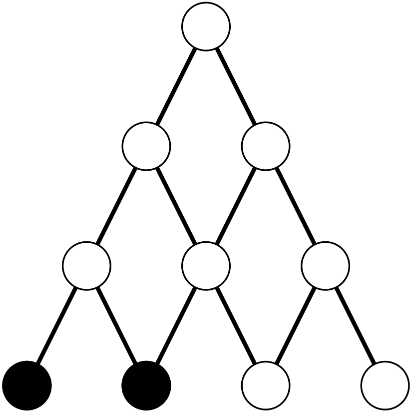
Rowmotion is the toggle group action that composes all the toggles from top to bottom.



Rowmotion

Toggle group actions are compositions of toggles.

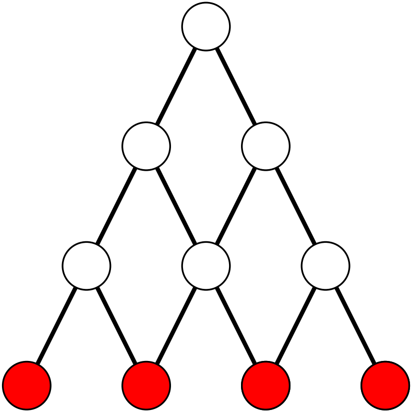
Rowmotion is the toggle group action that composes all the toggles from top to bottom.



Rowmotion

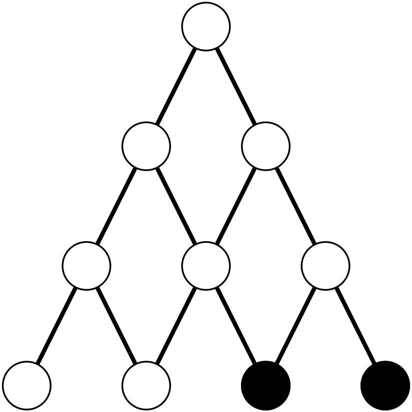
Toggle group actions are compositions of toggles.

Rowmotion is the toggle group action that composes all the toggles from top to bottom.



Rowmotion

Toggle group actions are compositions of toggles.
Rowmotion is the toggle group action that composes all the toggles from top to bottom.



The conjugacy of rowmotion, promotion, and gyration

Theorem (N. Williams and S. 2012)

In any ranked poset, there are equivariant bijections between the order ideals under under rowmotion (toggle top to bottom), promotion (toggle left to right), and gyration (toggle evens then odds).

Rowmotion, promotion, and gyration
all have the same orbit structure!

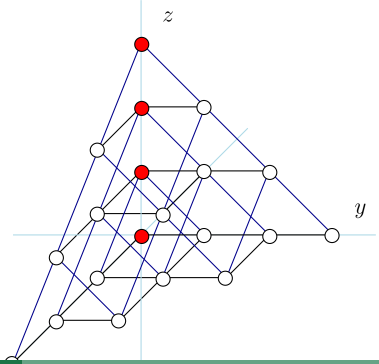
Gyration as a toggle group action

Theorem (N. Williams and S. 2012)

Gyration on fully-packed loops has the same orbit structure as rowmotion on order ideals of the ASM poset.

Corollary

Rowmotion on the ASM poset exhibits resonance with frequency $2n$.



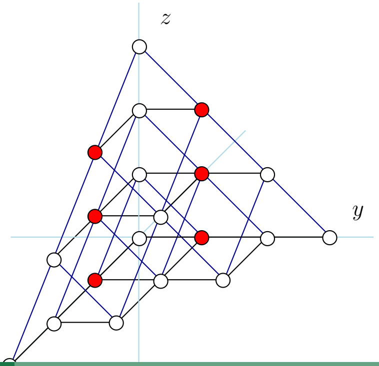
Gyration as a toggle group action

Theorem (N. Williams and S. 2012)

Gyration on fully-packed loops has the same orbit structure as rowmotion on order ideals of the ASM poset.

Corollary

Rowmotion on the ASM poset exhibits resonance with frequency $2n$.



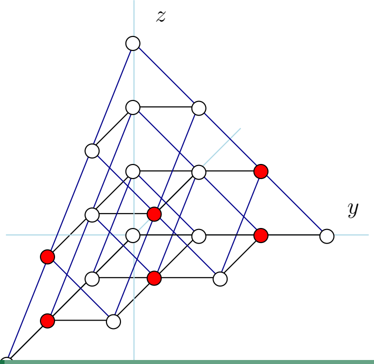
Gyration as a toggle group action

Theorem (N. Williams and S. 2012)

Gyration on fully-packed loops has the same orbit structure as rowmotion on order ideals of the ASM poset.

Corollary

Rowmotion on the ASM poset exhibits resonance with frequency $2n$.



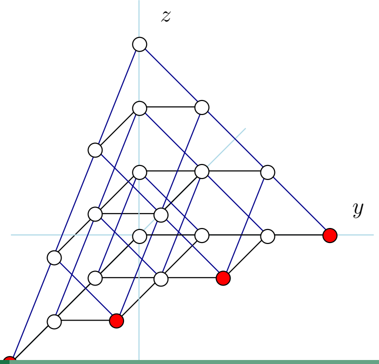
Gyration as a toggle group action

Theorem (N. Williams and S. 2012)

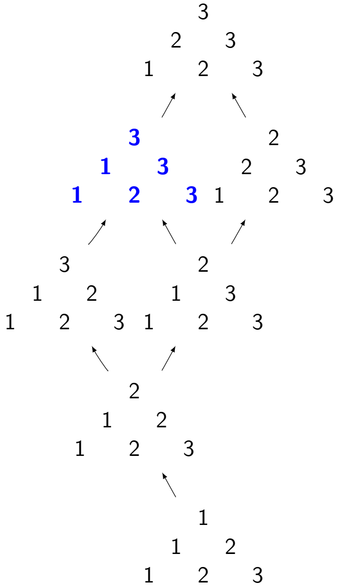
Gyration on fully-packed loops has the same orbit structure as rowmotion on order ideals of the ASM poset.

Corollary

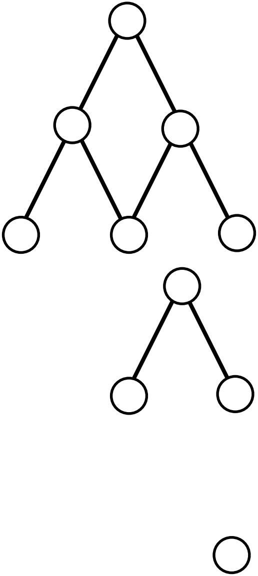
Rowmotion on the ASM poset exhibits resonance with frequency $2n$.



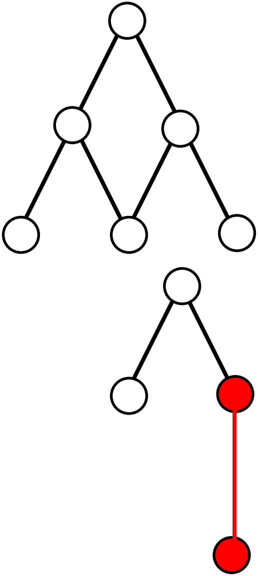
A partial order on TSSCPP



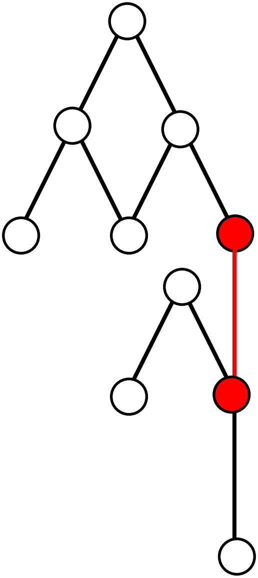
TSSCPP tetrahedral poset



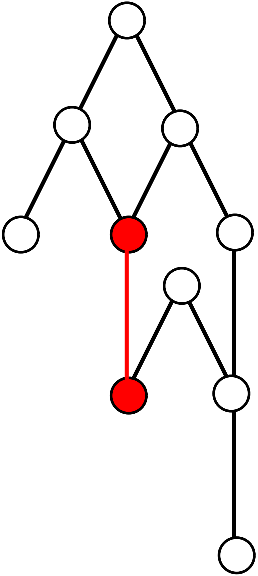
TSSCPP tetrahedral poset



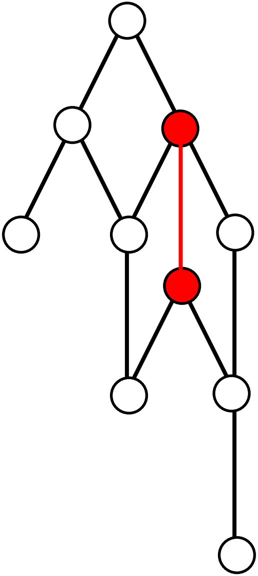
TSSCPP tetrahedral poset



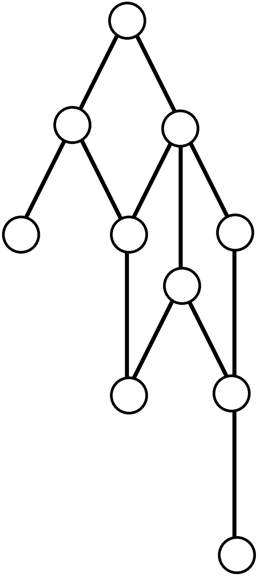
TSSCPP tetrahedral poset



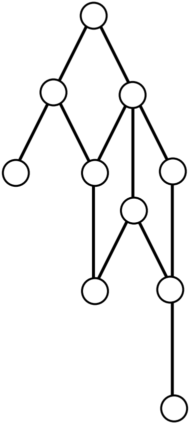
TSSCPP tetrahedral poset



TSSCPP tetrahedral poset

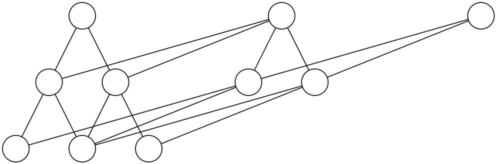


TSSCPP poset

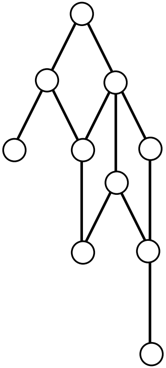


TSSCPPs inside a $2n \times 2n \times 2n$ box are in bijection with order ideals in this $(n-1)$-layer poset.

ASM and TSSCPP posets



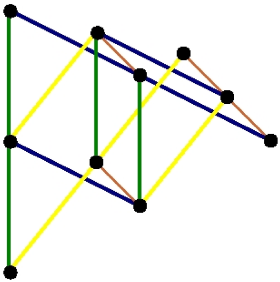
ASM



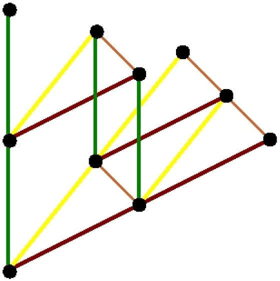
TSSCPP

ASM and TSSCPP tetrahedral posets (S. 2011)

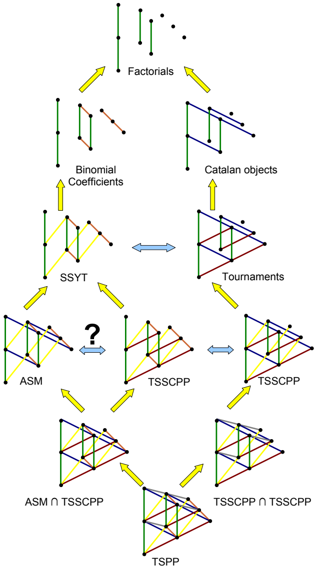
ASM



TSSCPP



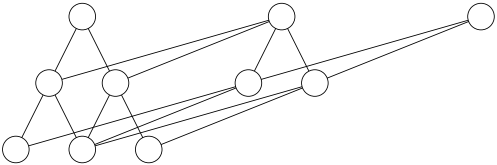
Tetrahedral poset family (S. 2011)



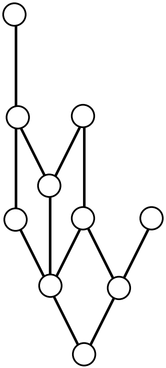
Dynamics: Rowmotion on the ASM and TSSCPP posets

Rowmotion exhibits resonance with frequency:

2n



ASM

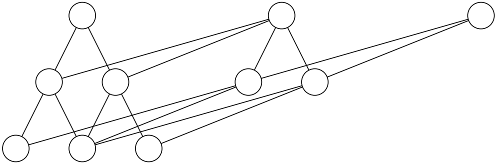


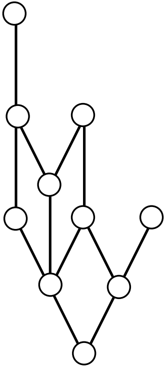
TSSCPP

Dynamics: Rowmotion on the ASM and TSSCPP posets

Rowmotion exhibits resonance with frequency:

2n **3n − 2**



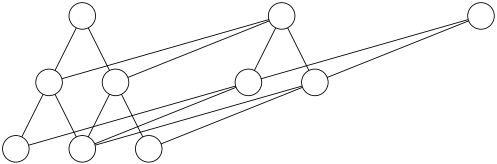


ASM TSSCPP

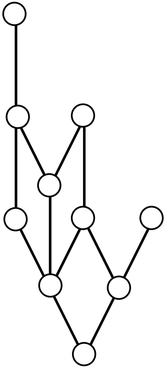
Dynamics: Rowmotion on the ASM and TSSCPP posets

Rowmotion exhibits resonance with frequency:

$\mathbf{2n}$

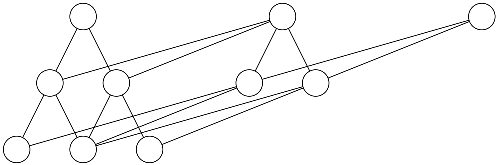


$(\mathbf{3n-2})^*$



Can we find a toggle action on the ASM poset similar to rowmotion on the TSSCPP poset?

ASM superpromotion



Toggle the largest layer from left to right, then the next largest layer, etc. For the orbit sizes, see [S. Williams 2012, Fig. 22]

Exercise

Compute the superpromotion orbit on an order ideal of the ASM poset of order 3. *Compute the rowmotion orbit of an order ideal of the TSSCPP poset of order* 3. *See that they both have size* 7.

Open Problem

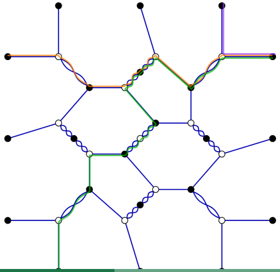
Show that TSSCPP rowmotion and ASM superpromotion both exhibit resonance with frequency $3n-2$. *(Find a projection to something that rotates with order* $3n-2$.*)*

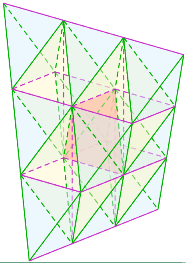
Pockets for web equivalence classes

The *dual diskoid* of an uncontracted 4-hourglass plabic graph G is the 2-dimensional simplicial complex $D(G)$ whose vertices are faces of G and edges correspond to adjacent faces.

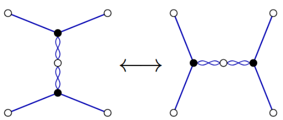
Definition (Gaetz, S., Swanson, Wu 2026+)

The *pocket* of a move-class $\mathcal{C}$ of fully reduced 4-hourglass plabic graphs is the simplicial complex $P(\mathcal{C})$ obtained by gluing together the dual diskoids $D(G)$ for $G \in \mathcal{C}$ by attaching tetrahedra between the portions involved in a move.

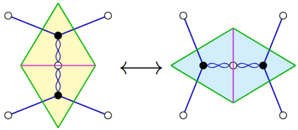




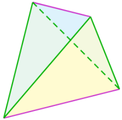
Pockets for web equivalence classes



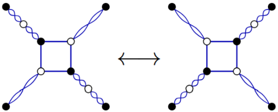
IH move between webs



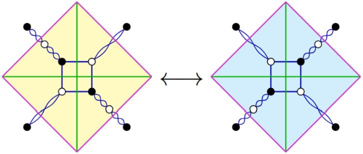
Dual diskoids



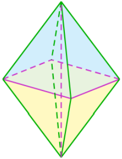
tetrahedron



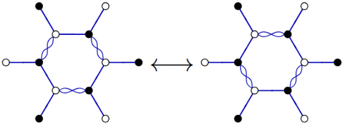
Square move between webs



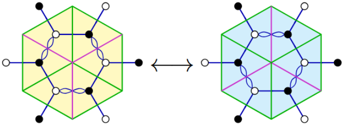
Dual diskoids



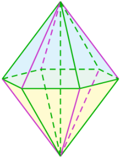
octahedron



Benzene move between webs



Dual diskoids



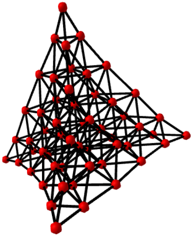
dodecahedron

Pockets for web equivalence classes

Theorem (Gaetz, S., Swanson, Wu 2026+)

The pocket $P(\mathcal{C})$ of a move-class $\mathcal{C}$

- *is a flag simplicial complex that is contractible, but not always homeomorphic to a ball,*
- *is embeddable in $\mathbb{R}^3$ if and only if its associated poset is a distributive lattice,*
- *has a separation labeling which allows us to define a natural notion of distance that has coherent geodesics.*



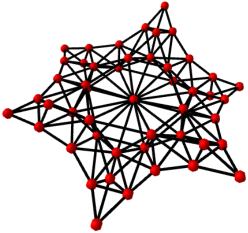
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

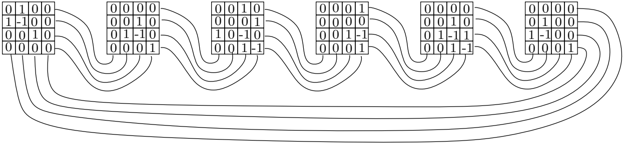
Pockets for web equivalence classes

Theorem (Gaetz, S., Swanson, Wu 2026+)

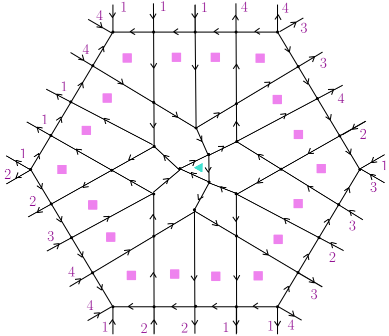
The pocket $P(\mathcal{C})$ of a move-class $\mathcal{C}$

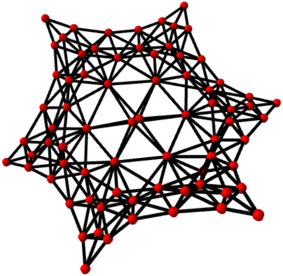
- *is a flag simplicial complex that is contractible, but not always homeomorphic to a ball,*
- *is embeddable in $\mathbb{R}^3$ if and only if its associated poset is a distributive lattice,*
- *has a separation labeling which allows us to define a natural notion of distance that has coherent geodesics.*

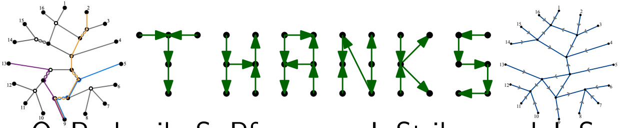




An ASM region surrounding a benzene face







- C. Gaetz, O. Pechenik, S. Pfannerer, J. Striker, and J. Swanson:
  - Rotation–invariant web bases from hourglass plabic graphs. *Inventiones Math.* **243** (2026) no.2, 102pp.
  - Promotion permutations for tableaux. *Comb. Theory*, **4** (2024) 55pp.
- K. Dilks, O. Pechenik, and J. Striker, Resonance in orbits of plane partitions and increasing tableaux. *JCTA* **148** (2017) 244–274.
- J. Striker and N. Williams, Promotion and rowmotion. *Eur. J. Combin.* **33** (2012), no. 8, 1919–1942.
- J. Striker, A unifying poset perspective on alternating sign matrices, plane partitions, Catalan objects, tournaments, and tableaux. *Adv. Appl. Math.* **46** (2011), no. 4, 583–609.
- J. Striker, The toggle group, homomesy, and the Razumov-Stroganov correspondence. *Electron. J. Combin.* **22** (2015), no. 2.
- C. Gaetz, J. Striker, J. Swanson, and H. Wu, Web bases, pocket complexes, and affine buildings, FPSAC 2026 extended abstract, full version on arXiv soon.