Wronskians, total positivity, and real Schubert calculus

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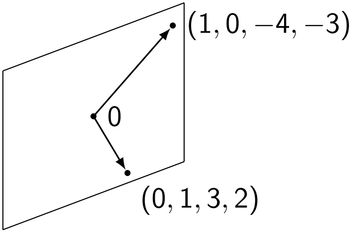
The Grassmannian and total positivity

- The *Grassmannian* $\mathrm{Gr}_{k,n}(\mathbb{R})$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

$V :=$



$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}(\mathbb{R})$$

$$\Delta_{12} = 1, \ \ \Delta_{13} = 3, \ \ \Delta_{14} = 2, \ \ \Delta_{23} = 4, \ \ \Delta_{24} = 3, \ \ \Delta_{34} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \dots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$ *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I, and *totally positive* if $\Delta_I(V) > 0$ for all k-subsets I.

Complete flag variety

- The *complete flag variety* $\mathrm{Fl}_n(\mathbb{R})$ consists of tuples of subspaces $(V_1, \ldots, V_{n-1})$ of $\mathbb{R}^n$, where

$$V_1 \subset \cdots \subset V_{n-1} \quad \text{and} \quad \dim(V_k) = k \text{ for all } 1 \leq k \leq n-1.$$

We say that $(V_1, \ldots, V_{n-1})$ is *totally nonnegative* if all its Plücker coordinates are nonnegative, i.e., V_k is totally nonnegative in $\mathrm{Gr}_{k,n}(\mathbb{R})$ for all $1 \leq k \leq n-1$. Similarly, we say that $(V_1, \ldots, V_{n-1})$ is *totally positive* if all its Plücker coordinates are positive.

- e.g. Let $n := 3$, and let $(V_1, V_2) \in \mathrm{Fl}_3(\mathbb{R})$ be given by the matrix

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{with} \quad \begin{array}{l} \Delta_1 = 1,\ \Delta_2 = a,\ \Delta_3 = b, \\ \Delta_{12} = 1,\ \Delta_{13} = c,\ \Delta_{23} = ac - b. \end{array}$$

Then (V_1, V_2) is totally positive if and only if

$$a,\ b,\ c,\ ac - b > 0.$$

The Wronskian

- The *Wronskian* of k linearly independent functions $f_1, \dots, f_k : \mathbb{R} \to \mathbb{R}$ is

$$\mathrm{Wr}(f_1, \dots, f_k) := \det \begin{bmatrix} f_1 & \cdots & f_k \\ f_1' & \cdots & f_k' \\ \vdots & \ddots & \vdots \\ f_1^{(k-1)} & \cdots & f_k^{(k-1)} \end{bmatrix}.$$

- e.g. $\mathrm{Wr}(f, g) = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - f'g = f^2(\frac{g}{f})'$.
- Let $V := \mathrm{span}(f_1, \dots, f_k)$. Then $\mathrm{Wr}(V)$ is well-defined up to a scalar. Its zeros are points in $\mathbb{R}$ where some nonzero $f \in V$ has a zero of order k.
- We identify $\mathbb{R}^n$ with the space of polynomials of degree at most $n-1$:

$$\mathbb{R}^n \leftrightarrow \mathbb{R}[x]_{\leq n-1}, \quad (a_1, \dots, a_n) \leftrightarrow a_1 + a_2 x + \cdots + a_n x^{n-1}.$$

We obtain the *Wronski map* $\mathrm{Wr} : \mathrm{Gr}_{k,n}(\mathbb{R}) \to \mathbb{P}(\mathbb{R}[x]_{\leq k(n-k)})$.

- e.g. Let $V := \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}(\mathbb{R})$. Then

$$\mathrm{Wr}(V) = \mathrm{Wr}(1 - 4x^2 - 3x^3, x + 3x^2 + 2x^3) = 1 + 6x + 10x^2 + 6x^3 + x^4.$$

Wronskians and total positivity

Theorem (Karp (2021))

(i) The complete flag $(V_1, \ldots, V_{n-1})$ *is totally nonnegative if and only if* $\mathrm{Wr}(V_k)$ *is nonzero on the interval* $(0, \infty)$*, for all* $1 \le k \le n-1$*.*
(ii) The complete flag $(V_1, \ldots, V_{n-1})$ *totally positive if and only if* $\mathrm{Wr}(V_k)$ *is nonzero on the interval* $[0, \infty]$*, for all* $1 \le k \le n-1$*.*

- e.g. Let $n := 3$, and let $(V_1, V_2) \in \mathrm{Fl}_3(\mathbb{R})$ be given by the matrix

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{with} \quad \begin{aligned} \mathrm{Wr}(V_1) &= 1 + ax + bx^2, \\ \mathrm{Wr}(V_2) &= 1 + 2cx + (ac - b)x^2. \end{aligned}$$

Part (ii) says that a, b, c, $ac - b > 0$ if and only if $\mathrm{Wr}(V_1)$ and $\mathrm{Wr}(V_2)$ are positive on $[0, \infty]$. The forward direction is immediate, and the reverse direction follows by calculation (but the general proof is topological).

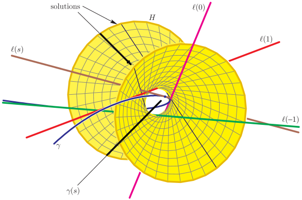
- The theorem also gives new total nonnegativity and total positivity tests for $\mathrm{Fl}_n(\mathbb{R})$ using the coefficients of the Wronskians.

Shapiro–Shapiro conjecture (1995)

- Schubert (1886): Let $W_1, \ldots, W_{k(n-k)} \in \mathrm{Gr}_{k,n}(\mathbb{C})$ be generic. Then there are $\frac{1!2!\cdots(k-1)!(k(n-k))!}{(n-k)!(n-k+1)!\cdots(n-1)!}$ elements $U \in \mathrm{Gr}_{n-k,n}(\mathbb{C})$ such that

$$U \cap W_i \neq \{0\} \qquad \text{for all } 1 \leq i \leq k(n-k).$$

- B. and M. Shapiro conjectured that if each W_i is an osculating plane to the *rational normal curve* $\gamma(x) := (1, x, \ldots, x^{n-1})$, then every U is real.
- e.g. $k = 2$, $n = 4$



F. Sottile, "Frontiers of reality in Schubert calculus"

- García-Puente, Hein, Hillar, Martín del Campo, Ruffo, Sottile, and Teitler (2012) made the more general *secant conjecture*: one can take each W_i to be spanned by k points of the form $\gamma(x)$, such that the values $x \in \mathbb{R}$ chosen for each W_i lie in $k(n-k)$ disjoint intervals.

Secant conjecture and Eremenko's conjecture

- The Shapiro–Shapiro conjecture can be reformulated as follows:

Theorem (Mukhin, Tarasov, Varchenko (2009))

Let $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$. If all complex zeros of $\mathrm{Wr}(V)$ are real, then V is real.

- e.g. Let $\mathrm{Wr}(V) := (x+a)^2(x+b)^2$. The two solutions $V \in \mathrm{Gr}_{2,4}(\mathbb{C})$ are

$$\big\langle (x+a)(x+b), x(x+a)(x+b) \big\rangle \quad \text{and} \quad \big\langle (x+a)^3, (x+b)^3 \big\rangle.$$

- The secant conjecture is still open. Eremenko (2015) showed that it is implied by the following conjecture:

Conjecture (Eremenko (2015))

Let $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$. If all complex zeros of $\mathrm{Wr}(V)$ are real, then every nonzero $f \in V$ has at most $k-1$ zeros in any interval of $\mathbb{R}$ on which $\mathrm{Wr}(V)$ is nonzero.

- The case $k = 2$ of both conjectures was proved by Eremenko, Gabrielov, Shapiro, and Vainshtein (2006).

Total positivity conjecture

Conjecture (Mukhin, Tarasov (2017); Karp (2021))

Let $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$.
(i) If all zeros of $\mathrm{Wr}(V)$ *lie in* $[-\infty, 0]$*, then* V *is totally nonnegative.*
(ii) If all zeros of $\mathrm{Wr}(V)$ *lie in* $(-\infty, 0)$*, then* V *is totally positive.*

- e.g. Let $\mathrm{Wr}(V) := (x+a)^2(x+b)^2$. If $a, b > 0$, then the two solutions

$$\begin{bmatrix} ab & a+b & 1 & 0 \\ 0 & ab & a+b & 1 \end{bmatrix} \text{ and } \begin{bmatrix} a^3 & 3a^2 & 3a & 1 \\ b^3 & 3b^2 & 3b & 1 \end{bmatrix} \text{ are totally positive.}$$

Theorem (Karp (2021))

The conjecture above is equivalent to Eremenko's conjecture. It implies a totally positive generalization of the secant conjecture.

- The proof uses the characterization of the totally positive part of $\mathrm{Fl}_n(\mathbb{R})$ presented earlier, along with classical results about Chebyshev systems and disconjugate linear differential equations.