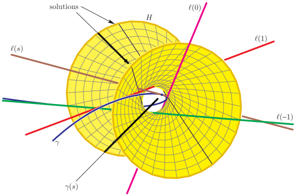
Total positivity in Schubert calculus

Slides available at `snkarp.github.io`



F. Sottile, "Frontiers of reality in Schubert calculus"

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`arXiv:2309.04645` (joint with Kevin Purbhoo)

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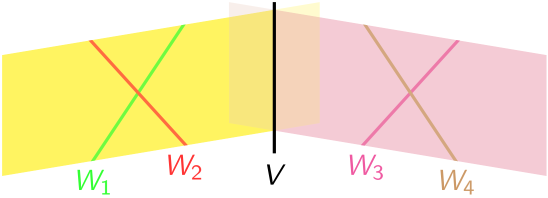
Schubert calculus (1886)

- Divisor Schubert problem: given subspaces $W_1, \ldots, W_{d(m-d)} \subseteq \mathbb{C}^m$ of dimension $m - d$, find all

$$d\text{-subspaces } V \subseteq \mathbb{C}^m \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- e.g. $d = 2$, $m = 4$ (projectivized). Given 4 lines $W_i \subseteq \mathbb{CP}^3$, find all lines $V \subseteq \mathbb{CP}^3$ intersecting all 4. Generically, there are 2 solutions.



1	2
3	4

1	3
2	4

We can see the 2 solutions explicitly when two pairs of the lines intersect.

- If the W_i's are generic, the number of solutions V is $f^{\square}$, the number of *standard Young tableaux* of rectangular shape $d \times (m - d)$.
- Fulton (1984): "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems."

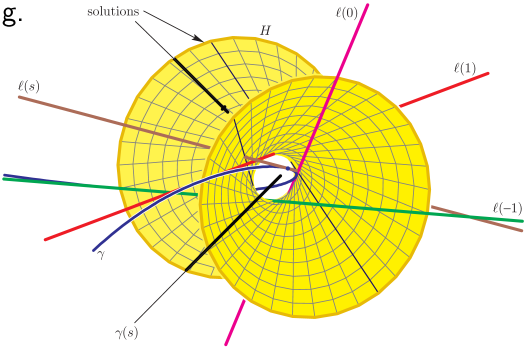
Shapiro–Shapiro conjecture

Shapiro–Shapiro conjecture (1993)

Let $W_1, \ldots, W_{d(m-d)} \subseteq \mathbb{R}^m$ be $(m-d)$-subspaces osculating the moment curve $\gamma(t) := (\frac{t^{m-1}}{(m-1)!}, \frac{t^{m-2}}{(m-2)!}, \ldots, t, 1)$ at real points. Then there exist $f^{\square}$

$$d\text{-subspaces } V \subseteq \mathbb{R}^m \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- e.g.



F. Sottile, "Frontiers of reality in Schubert calculus"

- This Schubert problem arises in the study of linear series in algebraic geometry, ordinary differential equations, and pole-placement problems in control theory.

Shapiro–Shapiro conjecture and secant conjecture

- Sottile (1999) tested the conjecture and proved it asymptotically.
- Eremenko–Gabrielov (2002): cases $d \leq 2$, $m - d \leq 2$.
- Mukhin–Tarasov–Varchenko (2009): full conjecture via the *Bethe ansatz*.
- Levinson–Purbhoo (2021): topological proof of the full conjecture.

Secant conjecture, divisor form (Sottile (2003))

Let $W_1, \dots, W_{d(m-d)} \subseteq \mathbb{R}^m$ be $(m-d)$-subspaces secant to the moment curve γ along non-overlapping intervals. Then there exist $f^{\square}$ real

$$d\text{-subspaces } V \subseteq \mathbb{R}^m \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- Eremenko–Gabrielov–Shapiro–Vainshtein (2006): case $m - d \leq 2$.

Theorem (Karp–Purbhoo (2023))

The divisor form of the secant conjecture is true.

Positive Shapiro–Shapiro conjecture

- The (projective) *Plücker coordinates* of a d-subspace $V \subseteq \mathbb{C}^m$ are the $d \times d$ minors $\Delta_J(V)$ of a $d \times m$ matrix whose rows form a basis of V.

- e.g. $V = \text{rowspan}\left(\begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix}\right) \rightsquigarrow$ $\begin{matrix} \Delta_{1,2}(V) = 1 & \Delta_{2,3}(V) = 4 \\ \Delta_{1,3}(V) = 3 & \Delta_{2,4}(V) = 3 \\ \Delta_{1,4}(V) = 2 & \Delta_{3,4}(V) = 1 \end{matrix}$

Positivity conjecture (Mukhin–Tarasov (2017); Karp (2021))

Let $W_1, \ldots, W_{d(m-d)} \subseteq \mathbb{R}^m$ be $(m-d)$-subspaces osculating the moment curve $\gamma(t)$ at real points $t_1, \ldots, t_{d(m-d)} > 0$. Then there exist $f^{\square}$ d-subspaces $V \subseteq \mathbb{R}^m$ with all $\Delta_J(V) > 0$ such that $V \cap W_i \neq \{0\}$ for all i.

- Karp (2023): the positivity conjecture is equivalent to a conjecture of Eremenko (2015), which implies the divisor form of the secant conjecture.

- Karp–Purbhoo (2023): the positivity conjecture is true. To prove it, we explicitly solve for all $\Delta_J(V)$ over $\mathbb{C}[\mathfrak{S}_{d(m-d)}]$.

Universal Plücker coordinates

- Shapiro–Shapiro problem: given $(m-d)$-subspaces $W_1, \ldots, W_{d(m-d)} \subseteq \mathbb{C}^m$ osculating $\gamma(t)$ at points $t_1, \ldots, t_{d(m-d)} \in \mathbb{C}$, find all

$$d\text{-subspaces } V \subseteq \mathbb{C}^m \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

Theorem (Karp–Purbhoo (2023))

There exist commuting linear operators $\beta_J = \beta_J(t_1, \ldots, t_{d(m-d)})$ indexed by d-subsets $J \subseteq \{1, \ldots, m\}$, satisfying:

(i) There is a bijection between the eigenspaces of the β_J's and the solutions V above, sending the eigenvalue of β_J to the $\Delta_J(V)$.

(ii) If $t_1, \ldots, t_{d(m-d)} > 0$, then the β_J's are positive definite.

$$\beta_J := \sum_{\substack{X \subseteq \{1,\ldots,d(m-d)\},\\ |X| = |\lambda(J)|}} \left(\prod_{i \notin X} t_i \right) \sum_{\pi \in \mathfrak{S}_X} \chi^{\lambda(J)}(\pi)\pi \in \mathbb{C}[\mathfrak{S}_{d(m-d)}].$$

- We show the β_J's satisfy the *Plücker relations* using the *KP hierarchy*.

New proof: higher Gaudin Hamiltonians

- The *higher Gaudin Hamiltonian* associated to the partition λ is

$$T_\lambda := (t_1 + \mathbf{d}_1) \cdots (t_n + \mathbf{d}_n) s_\lambda(h) \in \operatorname{End}\big((\mathbb{C}^d)^{\otimes n}\big),$$

where:

- h is a $d \times d$ matrix;
- $s_\lambda(h)$ is the Schur polynomial evaluated at the eigenvalues of h; and
- $\mathbf{d}_i$ is the derivative with respect to h^{T} acting in the ith tensor factor.

Theorem (Alexandrov–Leurent–Tsuboi–Zabrodin (2014))

The T_λ's pairwise commute and satisfy the Plücker relations.

Theorem (Karp–Mukhin–Tarasov (2024))

(i) We have $\beta_J = T_{\lambda(J)}|_{h=0}$ when $n = d(m-d)$.

(ii) If $t_1, \ldots, t_n > 0$ and h is positive definite, then so is T_λ.

Future directions

- Further explore the connection to the KP hierarchy.
- What happens to the higher Gaudin Hamiltonian T_λ if s_λ is replaced by a different symmetric function?
- Address generalizations and variations of the Shapiro–Shapiro conjecture: the discriminant conjecture, the general form of the secant conjecture, the monotone conjecture, the total reality conjecture for convex curves, . . .

Thank you!