Sign variation, the Grassmannian, and total positivity

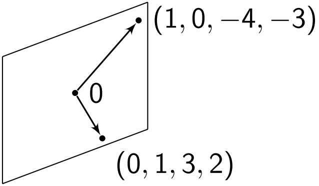
arXiv:1503.05622
Slides available at math.berkeley.edu/~skarp

Steven N. Karp, UC Berkeley

FPSAC 2015
KAIST, Daejeon

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces V of $\mathbb{R}^n$.

$V :=$ 

$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}$$

$$\Delta_{\{1,2\}} = 1, \Delta_{\{1,3\}} = 3, \Delta_{\{1,4\}} = 2, \Delta_{\{2,3\}} = 4, \Delta_{\{2,4\}} = 3, \Delta_{\{3,4\}} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for $I \in \binom{[n]}{k}$ let $\Delta_I(V)$ be the $k \times k$ minor of V with columns I. The *Plücker coordinates* $\Delta_I(V)$ are well-defined up to multiplication by a global nonzero constant.
- We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if $\Delta_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$. Denote the set of such V by $\mathrm{Gr}_{k,n}^{\geq 0}$, called the *totally nonnegative Grassmannian*.

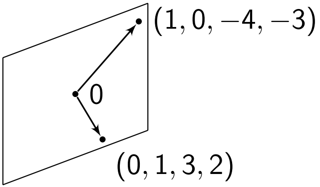
Sign variation

- For $x \in \mathbb{R}^n$, let $\mathrm{var}(x)$ be the number of sign changes in the sequence $x_1, x_2, \cdots, x_n$, ignoring any zeros. (We define $\mathrm{var}(0) := -1$.)

$$\mathrm{var}((1, -4, 0, -3, 6, 0, -1)) = \mathrm{var}((1, -4, -3, 6, -1)) = 3$$

Theorem (Gantmakher, Krein (1950); Schoenberg, Whitney (1951))

Let $V \in \mathrm{Gr}_{k,n}$. Then V is totally nonnegative iff $\mathrm{var}(x) \le k-1$ for all $x \in V$.

- e.g. $V :=$  $\in \mathrm{Gr}_{2,4}^{\ge 0}$.

- Note that every $V \in \mathrm{Gr}_{k,n}$ contains a vector x with $\mathrm{var}(x) \ge k-1$. So, the totally nonnegative subspaces are those whose vectors change sign as few times as possible.

A history of total positivity

- Pólya (1912) asked which linear $A : \mathbb{R}^k \to \mathbb{R}^n$ satisfy $\mathrm{var}(A(x)) \leq \mathrm{var}(x)$ for all $x \in \mathbb{R}^k$. Schoenberg (1930) showed that for injective A, this holds iff for $j = 1, \cdots, k$, all nonzero $j \times j$ minors of A have the same sign.

  formations. The problem of characterizing such transformations was attacked by Schoenberg in 1930 with only partial success

- Gantmakher, Krein (1935): The eigenvalues of a *totally positive* square matrix (all whose minors are positive) are real, positive, and distinct.
- Gantmakher, Krein (1950): *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, 359pp.
- Whitney (1952): The $n \times n$ totally positive matrices are dense in the $n \times n$ totally nonnegative matrices.
- Aissen, Schoenberg, Whitney (1952): Let $r_1, \cdots, r_n \in \mathbb{C}$. Then $r_1, \cdots, r_n$ are all nonnegative reals iff $s_\lambda(r_1, \cdots, r_n) \geq 0$ for all partitions λ.
- Karlin (1968): *Total Positivity, Volume I*, 576pp.
- Lusztig (1994) developed a theory of total positivity for G and G/P.
- Fomin and Zelevinsky (2000s) defined cluster algebras.
- Postnikov (2006) studied $\mathrm{Gr}_{k,n}^{\geq 0}$ from a combinatorial perspective.

How close is a subspace to being totally nonnegative?

- Can we determine $\max_{x \in V} \operatorname{var}(x)$ from the Plücker coordinates of V?

Theorem (Karp (2015))

Let $V \in \mathrm{Gr}_{k,n}$ and $m \geq k-1$.
(i) If $\operatorname{var}(x) \leq m$ for all $x \in V$, then

$$\operatorname{var}((\Delta_{J \cup \{i\}}(V))_{i \notin J}) \leq m - k + 1 \qquad \text{for all } J \in \binom{[n]}{k-1}.$$

The converse holds if V is generic (i.e. $\Delta_I(V) \neq 0$ for all I).
(ii) We can perturb V into a generic W with $\max_{x \in V} \operatorname{var}(x) = \max_{x \in W} \operatorname{var}(x)$.

- e.g. Let $V := \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}_{2,4}$ and $m := 2$. The fact that $\operatorname{var}(x) \leq 2$ for all $x \in V$ is equivalent to the fact that the 4 sequences

$$(\Delta_{\{1,2\}}, \Delta_{\{1,3\}}, \Delta_{\{1,4\}}) = (2,1,1), \quad (\Delta_{\{1,3\}}, \Delta_{\{2,3\}}, \Delta_{\{3,4\}}) = (1,4,-6),$$

$$(\Delta_{\{1,2\}}, \Delta_{\{2,3\}}, \Delta_{\{2,4\}}) = (2,4,-8), \quad (\Delta_{\{1,4\}}, \Delta_{\{2,4\}}, \Delta_{\{3,4\}}) = (1,-8,-6)$$

each change sign at most once.

How close is a subspace to being totally nonnegative?

- Can we determine $\max_{x \in V} \mathrm{var}(x)$ from the Plücker coordinates of V?

Theorem (Karp (2015))

Let $V \in \mathrm{Gr}_{k,n}$ and $m \geq k - 1$.
(i) If $\mathrm{var}(x) \leq m$ for all $x \in V$, then

$$\mathrm{var}((\Delta_{J \cup \{i\}}(V))_{i \notin J}) \leq m - k + 1 \qquad \textit{for all } J \in \binom{[n]}{k-1}.$$

The converse holds if V is generic (i.e. $\Delta_I(V) \neq 0$ for all I).
(ii) We can perturb V into a generic W with $\max_{x \in V} \mathrm{var}(x) = \max_{x \in W} \mathrm{var}(x)$.

- e.g. Consider

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \overset{0.1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0.1 & 1 \end{bmatrix} \overset{0.01}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & 0.01 \\ 0 & 1 & 0.1 & 1.001 \end{bmatrix}.$$

The 4 sequences of Plücker coordinates are

$$(\Delta_{\{1,2\}}, \Delta_{\{1,3\}}, \Delta_{\{1,4\}}) = (1, \overset{0.1}{\cancel{0}}, \overset{1.001}{\cancel{1}}), \quad (\Delta_{\{1,3\}}, \Delta_{\{2,3\}}, \Delta_{\{3,4\}}) = (\overset{0.1}{\cancel{0}}, -1, 1),$$

$$(\Delta_{\{1,2\}}, \Delta_{\{2,3\}}, \Delta_{\{2,4\}}) = (1, -1, \overset{-0.01}{\cancel{0}}), \quad (\Delta_{\{1,4\}}, \Delta_{\{2,4\}}, \Delta_{\{3,4\}}) = (\overset{1.001}{\cancel{1}}, \overset{-0.01}{\cancel{0}}, 1).$$

The totally positive Grassmannian

- We say that $V \in \mathrm{Gr}_{k,n}$ is *totally positive* if $\Delta_I(V) > 0$ for all $I \in \binom{[n]}{k}$.
- For $x \in \mathbb{R}^n$, let $\overline{\mathrm{var}}(x)$ be the maximum of $\mathrm{var}(y)$ over all $y \in \mathbb{R}^n$ obtained from x by changing zero components of x.

$$\overline{\mathrm{var}}((1, -4, 0, -3, 6, 0, -1)) = 5$$

Theorem (Gantmakher, Krein (1950))

$V \in \mathrm{Gr}_{k,n}$ is totally positive iff $\overline{\mathrm{var}}(x) \le k-1$ for all nonzero $x \in V$.

Theorem (Karp (2015))

Let $V \in \mathrm{Gr}_{k,n}$ and $m \ge k-1$. Then $\overline{\mathrm{var}}(x) \le m$ for all nonzero $x \in V$ iff

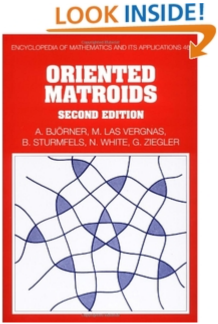
$$\overline{\mathrm{var}}((\Delta_{J \cup \{i\}}(V))_{i \notin J}) \le m - k + 1$$

for all $J \in \binom{[n]}{k-1}$ such that $\Delta_{J \cup \{i\}}(V) \ne 0$ for some i.

- Note that var is *increasing* while $\overline{\mathrm{var}}$ is *decreasing* with respect to genericity.

Oriented matroids

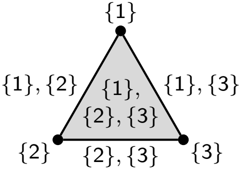
- An *oriented matroid* is a combinatorial abstraction of a real subspace, which records the Plücker coordinates up to sign, or equivalently the vectors up to sign.



- These results generalize to oriented matroids.

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

- Given $V \in \mathrm{Gr}_{k,n}$, let $M(V) := \{I \in \binom{[n]}{k} : \Delta_I(V) \neq 0\}$, called the *matroid* of V. The *matroid stratification* of $\mathrm{Gr}_{k,n}^{\geq 0}$ is a CW-decomposition.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$ 

- How can we find the cell of V (i.e. $M(V)$) in $\mathrm{Gr}_{k,n}^{\geq 0}$ using sign patterns?

Exercise

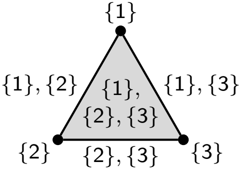
Let $V \in \mathrm{Gr}_{k,n}$ and $I \in \binom{[n]}{k}$. Then $\Delta_I(V) \neq 0$ iff V realizes all 2^k sign patterns in $\{+,-\}^k$ on I.

- Moreover, given $\omega \in \{+,-\}^k$, there exists $V \in \mathrm{Gr}_{k,n}$ which realizes all 2^k sign patterns in $\{+,-\}^k$ on I except for $\pm\omega$ (assuming $n > k$).

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

- Given $V \in \mathrm{Gr}_{k,n}$, let $M(V) := \{I \in \binom{[n]}{k} : \Delta_I(V) \neq 0\}$, called the *matroid* of V. The *matroid stratification* of $\mathrm{Gr}_{k,n}^{\geq 0}$ is a CW-decomposition.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$



- How can we find the cell of V (i.e. $M(V)$) in $\mathrm{Gr}_{k,n}^{\geq 0}$ using sign patterns?

Theorem (Karp (2015))

Let $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ and $I \in \binom{[n]}{k}$. Then $\Delta_I(V) \neq 0$ iff V realizes the following k sign patterns on I:
$(+,-,+,-,+,-,\cdots),(+,+,-,+,-,+,\cdots),(+,-,-,+,-,+,\cdots),\cdots$.

- Compare this to the fact that the matroid stratification of $\mathrm{Gr}_{k,n}^{\geq 0}$ is the refinement of n cyclically shifted *Schubert stratifications* (vs. all $n!$).

Further directions

- Is there an efficient way to test whether a given $V \in \mathrm{Gr}_{k,n}$ is totally positive using the data of sign patterns? (For Plücker coordinates, in order to test whether V is totally positive, we only need to check that some particular $k(n-k)$ Plücker coordinates are positive, not all $\binom{n}{k}$.)
- Is there a simple way to index the cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ using the data of sign patterns?
- Is there a nice stratification of the subset of the Grassmannian

$$\{V \in \mathrm{Gr}_{k,n} : \mathrm{var}(x) \leq m \text{ for all } x \in V\},$$

for fixed m? (If $m = k-1$, this is $\mathrm{Gr}_{k,n}^{\geq 0}$.)

Thank you!