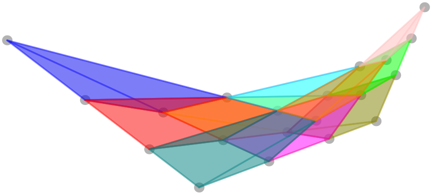
The $m = 1$ amplituhedron and cyclic hyperplane arrangements

`arXiv:1608.08288`

Slides available at `math.berkeley.edu/~skarp`



Steven N. Karp, UC Berkeley

joint work with Lauren Williams

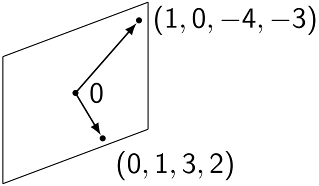
FPSAC 2017

Queen Mary University of London

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces V of $\mathbb{R}^n$.

$V :=$



$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

$$\Delta_{\{1,2\}} = 1,\ \Delta_{\{1,3\}} = 3,\ \Delta_{\{1,4\}} = 2,\ \Delta_{\{2,3\}} = 4,\ \Delta_{\{2,4\}} = 3,\ \Delta_{\{3,4\}} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \dots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all totally nonnegative V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.

Sign variation

- For $v \in \mathbb{R}^n$, let $\operatorname{var}(v)$ be the number of sign changes in the sequence $(v_1, v_2, \ldots, v_n)$, ignoring any zeros.

$$\operatorname{var}(1, -4, 0, -3, 6, 0, -1) = \operatorname{var}(1, -4, -3, 6, -1) = 3$$

Similarly, let $\overline{\operatorname{var}}(v)$ be the maximum number of sign changes we can get if we choose a sign for each zero component of v.

$$\overline{\operatorname{var}}(1, -4, 0, -3, 6, 0, -1) = 5$$

Theorem (Gantmakher, Krein (1950))

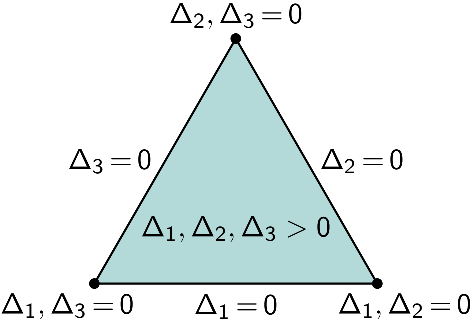
Let $V \in \operatorname{Gr}_{k,n}$. The following are equivalent:
(i) V is totally nonnegative;
(ii) $\operatorname{var}(v) \le k - 1$ for all $v \in V$;
(iii) $\overline{\operatorname{var}}(w) \ge k$ for all $w \in V^{\perp}$.

- e.g. $\begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \operatorname{Gr}_{2,4}^{\ge 0}$.
- The upper bound $k-1$ and the lower bound k are both 'best possible'.

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition. Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

$\mathrm{Gr}_{1,3}^{\geq 0} = \mathbb{P}_{\geq 0}^{2} \cong$



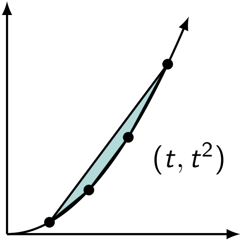
- $\mathrm{Gr}_{1,n}^{\geq 0}$ is an $(n-1)$-dimensional simplex in $\mathbb{P}^{n-1}$. So, one can think of the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ as a generalization of a simplex.

Cyclic hyperplane arrangements

- A *cyclic polytope* is a polytope (up to combinatorial equivalence) whose vertices line on the *moment curve* in $\mathbb{R}^k$

$$(t, t^2, \ldots, t^k) \qquad (t > 0).$$

- e.g. $k = 2$

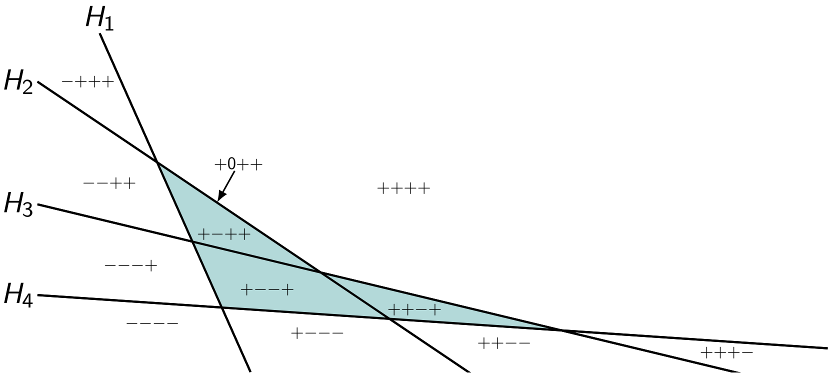


- Cyclic polytopes achieve the upper bound in the *upper bound theorem* of McMullen and Stanley.
- A *cyclic hyperplane arrangement* consists of hyperplanes in $\mathbb{R}^k$ of the form

$$tx_1 + t^2x_2 + \cdots + t^kx_k + 1 = 0 \qquad (t > 0).$$

Faces of cyclic hyperplane arrangements

- e.g.

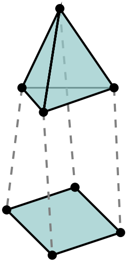


Theorem (Karp, Williams)

Let $\mathcal{H}$ be a cyclic hyperplane arrangement of n hyperplanes in $\mathbb{R}^k$.
(i) The bounded faces of $\mathcal{H}$ are labeled precisely by those sign vectors $\sigma \in \{0, +, -\}^n$ (up to sign) with $\overline{\mathrm{var}}(\sigma) = k$.
(ii) The unbounded faces of $\mathcal{H}$ are labeled precisely by σ with $\overline{\mathrm{var}}(\sigma) < k$.

Grassmann polytopes

- By definition, a polytope is the image of a simplex under an affine map:



simplex in $\mathbb{P}^{n-1}$ - - - - → $\mathrm{Gr}_{k,n}^{\geq 0}$

↓ linear map $\mathbb{R}^n \to \mathbb{R}^{m+1}$ ↓ linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$

polytope in $\mathbb{P}^m$ - - - - → *Grassmann polytope* in $\mathrm{Gr}_{k,k+m}$

A *Grassmann polytope* is the image of a map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

- When the matrix Z has positive maximal minors, the corresponding Grassmann polytope is called an *amplituhedron*, denoted $\mathcal{A}_{n,k,m}(Z)$.
- Amplituhedra are a common generalization of cyclic polytopes ($k = 1$) and totally nonnegative Grassmannians ($k + m = n$). They were introduced by Arkani-Hamed and Trnka in their study of *scattering amplitudes*.

The amplituhedron

Conjecture (Arkani-Hamed, Trnka (2014))

The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is 'triangulated' by the images of certain $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, coming from the BCFW recursion.

- This conjecture appears to be difficult, so we first considered $m = 1$.

Lemma

Let $W \in \mathrm{Gr}_{k+m,n}$ denote the subspace spanned by the rows of Z. Then

$$\mathcal{A}_{n,k,m}(Z) \cong \mathcal{B}_{n,k,m}(W) := \{V^{\perp} \cap W : V \in \mathrm{Gr}_{k,n}^{\geq 0}\} \subseteq \mathrm{Gr}_m(W).$$

- Using results of Gantmakher and Krein, we obtain

$$\mathcal{B}_{n,k,m}(W) \subseteq \{X \in \mathrm{Gr}_m(W) : k \leq \overline{\mathrm{var}}(v) \leq k+m-1 \text{ for all } v \in X \setminus \{0\}\}.$$

Problem

Does equality hold above?

The $m = 1$ amplituhedron

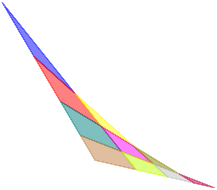
- We showed that equality does hold when $m = 1$:

$$\mathcal{B}_{n,k,1}(W) = \{w \in \mathbb{P}(W) : \overline{\text{var}}(w) = k\} \subseteq \mathbb{P}(W).$$

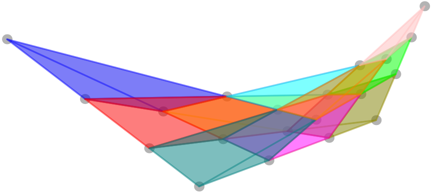
Theorem (Karp, Williams)

(i) $\mathcal{A}_{n,k,1}(Z)$ is isomorphic to the complex of bounded faces of a cyclic hyperplane arrangement of n hyperplanes in $\mathbb{R}^k$.
(ii) $\mathcal{A}_{n,k,1}(Z)$ is isomorphic to a subcomplex of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$.
(iii) $\mathcal{A}_{n,k,1}(Z)$ is homeomorphic to a closed ball of dimension k.

- Part (iii) follows directly from part (i) by a general result of Dong.



$\mathcal{A}_{6,2,1}$



$\mathcal{A}_{6,3,1}$