The totally nonnegative Grassmannian is a ball

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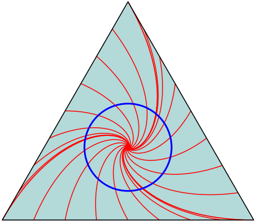
Slides available at www-personal.umich.edu/~snkarp



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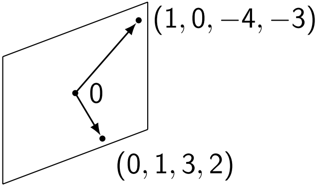
Dartmouth College

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

$V :=$



$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

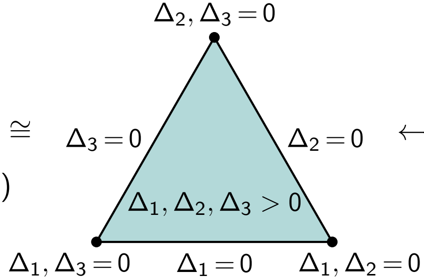
$$\Delta_{\{1,2\}} = 1,\ \Delta_{\{1,3\}} = 3,\ \Delta_{\{1,4\}} = 2,\ \Delta_{\{2,3\}} = 4,\ \Delta_{\{2,4\}} = 3,\ \Delta_{\{3,4\}} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \cdots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call $V \in \mathrm{Gr}_{k,n}$ *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all such V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.
- We can think of $\mathrm{Gr}_{k,n}^{\geq 0}$ as the generalization of a simplex into the Grassmannian.

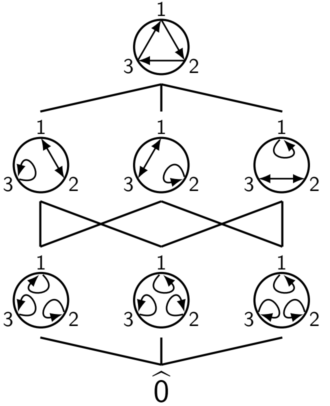
The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition due to Rietsch (1998) and Postnikov (2007). Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$ $(= \mathbb{P}_{\geq 0}^2)$



$\longleftrightarrow$

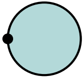
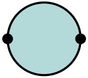


- Postnikov showed that the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is given by *circular Bruhat order* on *decorated permutations* with k *anti-excedances*.

The topology of $\mathrm{Gr}_{k,n}^{\geq 0}$

Conjecture (Postnikov (2007))

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ is a regular CW complex homeomorphic to a ball. That is, the closure of every cell is homeomorphic to a closed ball.

- e.g.  non-regular CW complex  regular CW complex

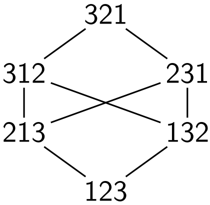
- Williams (2007): The face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is *thin* and *shellable*. Thus it is the face poset of *some* regular CW complex homeomorphic to a ball.
- Postnikov, Speyer, Williams (2009): $\mathrm{Gr}_{k,n}^{\geq 0}$ is a CW complex (via *matching polytopes* of plabic graphs).
- Rietsch, Williams (2010): $\mathrm{Gr}_{k,n}^{\geq 0}$ is a regular CW complex up to homotopy (via discrete Morse theory).

Theorem (Galashin, Karp, Lam (2017+))

$\mathrm{Gr}_{k,n}^{\geq 0}$ *is homeomorphic to a closed ball of dimension* $k(n-k)$.

Motivation 1: combinatorics of regular CW complexes

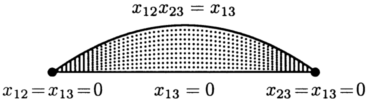
- Every convex polytope (decomposed into its open faces) is a regular CW complex. We can think of a regular CW complex as the 'next best thing' to a convex polytope.

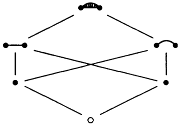


$\mathfrak{S}_3$

$\leftsquigarrow$

$$Y_3 = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : \begin{array}{l} x + z = 1, \\ \text{all minors} \geq 0 \end{array} \right\}$$

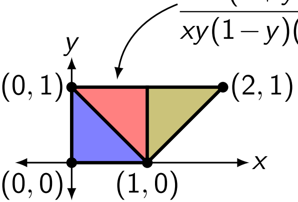




- Note that $\mathfrak{S}_n$ is not the face poset of a polytope. However, $\mathfrak{S}_n$ is shellable due to Edelman (1981), so it is the face poset of a regular CW complex homeomorphic to a ball, by work of Björner (1984).
- Bernstein: Is there a 'naturally occurring' such regular CW complex?
- Fomin and Shapiro (2000) conjectured that $Y_n \subseteq \mathrm{SL}_n$ is such a regular CW complex. This was proved by Hersh (2014), in general Lie type.

Motivation 2: positive geometries and physics

- Arkani-Hamed, Bai, Lam (2017): a *positive geometry* is a space equipped with a canonical differential form, which has logarithmic singularities at the boundaries of the space. Examples include convex polytopes.



$$\frac{(1+y)dxdy}{xy(1-y)(1-x+y)} = \frac{dxdy}{xy(1-x-y)} + \frac{dxdy}{(1-x)(1-y)(x+y-1)} + \frac{dxdy}{(x-1)(1-y)(1-x+y)}$$

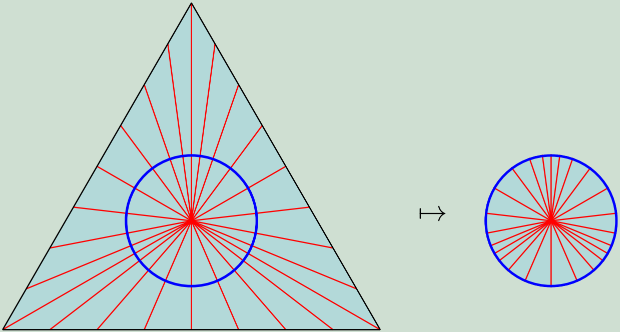
- The canonical differential forms of several positive geometries have already been interpreted physically, including *amplituhedra* (simultaneous generalizations of cyclic polytopes and totally nonnegative Grassmannians), *associahedra*, and *cosmological polytopes* (work of Arkani-Hamed, Bai, Benincasa, He, Postnikov, Trnka, Yan, Zhang, and Zhang).
- Different triangulations of a positive geometry give different expressions for the form. The boundary structure is related to *locality*.

Showing a compact, convex set is homeomorphic to a ball

Theorem

Every compact, convex subset of $\mathbb{R}^d$ is homeomorphic to a closed ball.

Proof



- How do we generalize this argument to $\mathrm{Gr}_{k,n}^{\geq 0}$?

Cyclic symmetry of $\mathrm{Gr}_{k,n}^{\geq 0}$

- Define the *(left) cyclic shift map* S on $\mathbb{R}^n$ by

$$S(v) := (v_2, v_3, \cdots, v_n, (-1)^{k-1} v_1) \quad \text{for } v = (v_1, \cdots, v_n) \in \mathbb{R}^n.$$

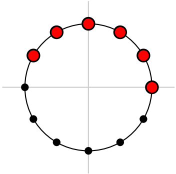
Then S acts on $\mathrm{Gr}_{k,n}$ as an automorphism of order n. This cyclic action preserves $\mathrm{Gr}_{k,n}^{\geq 0}$, and gives the 'cyclic symmetry' of the cell decomposition.

$$\begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \overset{S}{\longmapsto} \begin{bmatrix} 1 & -1 & -1 & -2 \\ 1 & 3 & 1 & 0 \end{bmatrix}$$

Theorem (Karp (2018+))

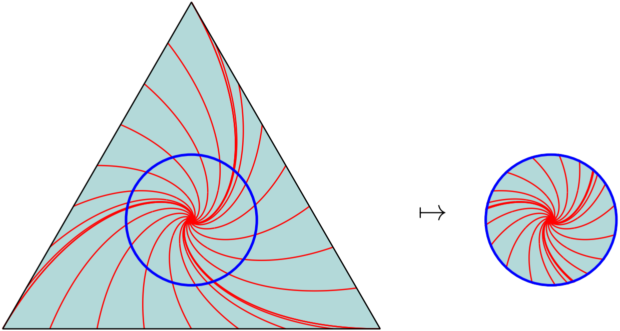
There exists a unique fixed point V_0 of the map S acting on $\mathrm{Gr}_{k,n}^{\geq 0}$.

- e.g. $V_0 = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \in \mathrm{Gr}_{2,6}^{\geq 0}$.



Showing $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a ball

- Now we regard S as a vector field on $\mathrm{Gr}_{k,n}$, which sends each $V \in \mathrm{Gr}_{k,n}$ along the trajectory $\exp(tS) \cdot V$ for $t \geq 0$. It turns out that this vector field contracts all of $\mathrm{Gr}_{k,n}^{\geq 0}$ to the fixed point V_0.
- e.g. $\mathrm{Gr}_{1,3}^{\geq 0}$

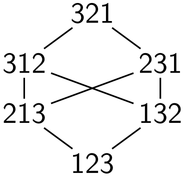


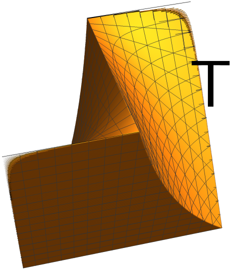
- We construct a homeomorphism to a closed ball as before. Along the way, we define global coordinates on $\mathrm{Gr}_{k,n}^{\geq 0}$ using the eigenvectors of S.

Other spaces homeomorphic to closed balls

- We also show that the following spaces are closed balls: the *cyclically symmetric* amplituhedron, Lam's compactification of the space of electrical networks, and Lusztig's totally nonnegative partial flag varieties $(G/P)_{\geq 0}$. It remains an open problem to show that all amplituhedra are closed balls.

- e.g. The totally nonnegative part of $G/P = \mathrm{Fl}_3$ consists of complete flags $0 \subset W_1 \subset W_2 \subset \mathbb{R}^3$ such that W_1 and W_2 are totally nonnegative subspaces. Its face poset is given by the *intervals* in $\mathfrak{S}_3$ (so there are 6 vertices, 8 edges, and 4 facets).





Thank you!