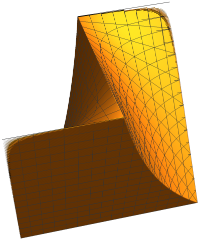
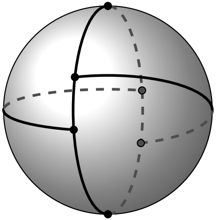
Regularity theorem for totally nonnegative flag varieties

Slides available at `lacim.uqam.ca/~snkarp`



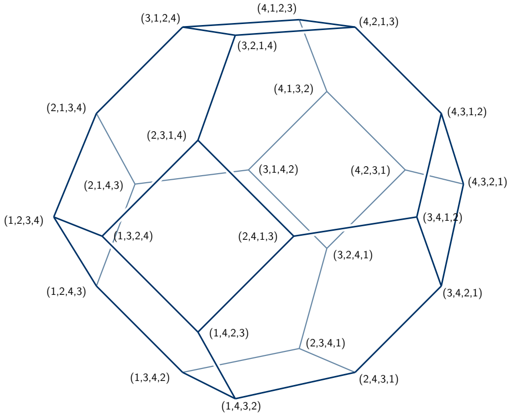


Steven N. Karp, LaCIM, Université du Québec à Montréal
joint work with Pavel Galashin and Thomas Lam

arXiv:1904.00527

FPSAC 2020

Permutohedron

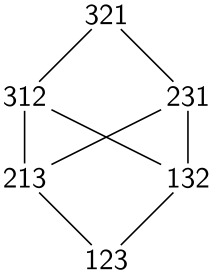


- The vertices of the permutohedron are $(\pi(1), \cdots, \pi(n)) \in \mathbb{R}^n$ for $\pi \in \mathfrak{S}_n$.
- The edges of the permutohedron are

$$(\cdots, i, \cdots, i+1, \cdots) \quad \longleftrightarrow \quad (\cdots, i+1, \cdots, i, \cdots).$$

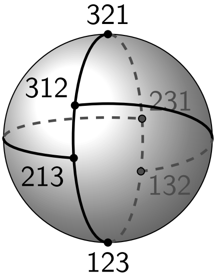
These correspond to cover relations in the *weak Bruhat order* on $\mathfrak{S}_n$.

Geometric realization of the strong Bruhat order



$\mathfrak{S}_3$ (strong order)

$\rightsquigarrow$



$\mathrm{Fl}_3^{\geq 0}$

- Using *total positivity*, we can define a space whose d-dimensional faces correspond to intervals of length d in the strong Bruhat order on $\mathfrak{S}_n$.
- This space is not a polytope! However, topologically it is just as good:
  1. it is partitioned into faces F, each homeomorphic to an open ball;
  2. the boundary ∂F of each face F is a union of lower-dimensional faces;
  3. the closure $\overline{F}$ of each face F is homeomorphic to a closed ball[1].

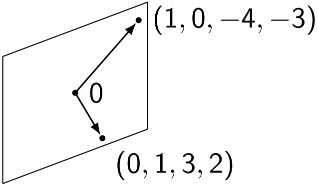
Such a space is called a *regular CW complex*.

[1] via a homeomorphism which sends F to the interior of the closed ball

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

$V :=$



$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

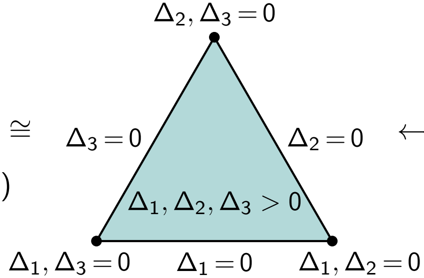
$$\Delta_{12} = 1, \;\; \Delta_{13} = 3, \;\; \Delta_{14} = 2, \;\; \Delta_{23} = 4, \;\; \Delta_{24} = 3, \;\; \Delta_{34} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \cdots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call $V \in \mathrm{Gr}_{k,n}$ *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all such V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.
- $\mathrm{Gr}_{1,n}$ is projective space $\mathbb{P}^{n-1}$, and its totally nonnegative part is a simplex. We can think of $\mathrm{Gr}_{k,n}^{\geq 0}$ as the Grassmannian notion of a simplex.

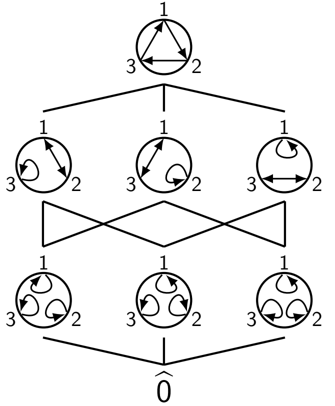
The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a decomposition into cells (open balls) due to Rietsch (1998) and Postnikov (2006). Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$ $(= \mathbb{P}^2_{\geq 0})$



$\longleftrightarrow$



- Postnikov showed that the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is given by *circular Bruhat order* on *decorated permutations* with k *anti-excedances*.

The topology of $\mathrm{Gr}_{k,n}^{\geq 0}$

Conjecture (Postnikov (2006))

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ *is a regular CW complex. Thus the closure of every cell is homeomorphic to a closed ball.*

- e.g. 

- Lusztig (1998): $\mathrm{Gr}_{k,n}^{\geq 0}$ is contractible.
- Williams (2007): The face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is graded, thin, and shellable.
- Postnikov, Speyer, Williams (2009): $\mathrm{Gr}_{k,n}^{\geq 0}$ is a CW complex.
- Rietsch, Williams (2010): Postnikov's conjecture is true up to homotopy.
- Galashin, Karp, Lam (2017): $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a closed ball.

Theorem (Galashin, Karp, Lam)

Postnikov's conjecture is true.

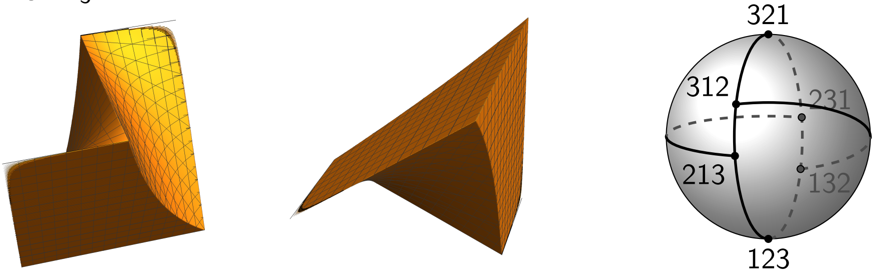
- Our result holds for all G/P, confirming a conjecture of Williams (2007).

The complete flag variety Fl_n

- Another instance of a partial flag variety G/P is the *complete flag variety* Fl_n, the set of tuples of subspaces

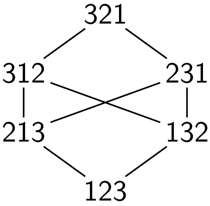
$$\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{R}^n, \qquad \text{where } V_k \in \mathrm{Gr}_{k,n} \text{ for all } k.$$

- Lusztig (1994): $\mathrm{Fl}_n^{\geq 0}$ is the subset where $V_k \in \mathrm{Gr}_{k,n}^{\geq 0}$ for all k.
- Lusztig (1994), Rietsch (1999): $\mathrm{Fl}_n^{\geq 0}$ has a cell decomposition whose d-dimensional cells are indexed by intervals of length d in $(\mathfrak{S}_n, \leq_{\text{strong}})$.
- e.g. $\mathrm{Fl}_3^{\geq 0}$



Motivation 1: combinatorics of regular CW complexes

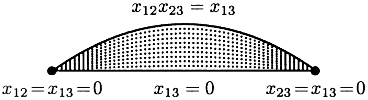
- Any convex polytope (decomposed into faces) is a regular CW complex.
- Björner (1984): Every regular CW complex is uniquely determined by its face poset (up to homeomorphism). Conversely, any poset which is *graded*, *thin*, and *shellable* is the face poset of some regular CW complex.

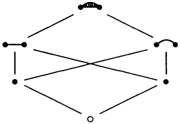


$\mathfrak{S}_3$ (strong order)

$$\mathrm{link}_{I_3}(U_3^{\geq 0}) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : \begin{array}{l} a + c = 1, \\ \text{all minors} \geq 0 \end{array} \right\}$$

$\leftrightsquigarrow$

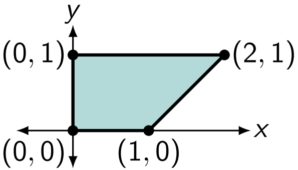




- Edelman (1981): $\mathfrak{S}_n$ is graded, thin, and shellable.
- Björner (1984): Is there a 'natural' regular CW complex with face poset $\mathfrak{S}_n$?
- Fomin and Shapiro (2000) conjectured that $\mathrm{link}_{I_n}(U_n^{\geq 0})$ is such a regular CW complex. This was proved by Hersh (2014), in general Lie type. We give a new proof of Hersh's theorem.

Motivation 2: scattering amplitudes and differential forms

- Arkani-Hamed, Bai, Lam (2017): we can associate to certain geometric spaces a *canonical differential form*.



$$\Big\updownarrow$$

$$\pm\frac{(1+y)dxdy}{xy(1-y)(1-x+y)}$$

$$\mathrm{Gr}_{2,4}^{\geq 0} = \left\{ \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{bmatrix} \right\}$$

$$\Big\updownarrow$$

$$\pm\frac{dxdydzdw}{\Delta_{12}\Delta_{23}\Delta_{34}\Delta_{14}}$$

- The differential form of the *amplituhedron*, a certain projection of $\mathrm{Gr}_{k,n}^{\geq 0}$, is conjecturally the tree-level scattering amplitude in planar $\mathcal{N} = 4$ SYM.
- Intuition from physics: the geometry determines the form, and vice-versa. In order to understand amplituhedra, first we need to understand $\mathrm{Gr}_{k,n}^{\geq 0}$.
- Other physically relevant spaces include *associahedra*, *accordiohedra*, ...

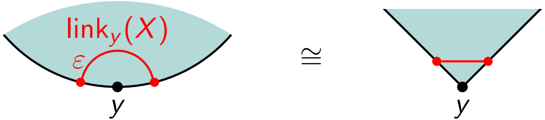
Proof: link induction

- We want to show that each cell closure $X \subseteq \mathrm{Gr}_{k,n}^{\geq 0}$ is a closed ball.

Theorem (consequence of generalized Poincaré conjecture)

Suppose that X is a compact topological manifold with boundary, whose interior X° is contractible and whose boundary ∂X is homeomorphic to a sphere. Then X is homeomorphic to a closed ball.

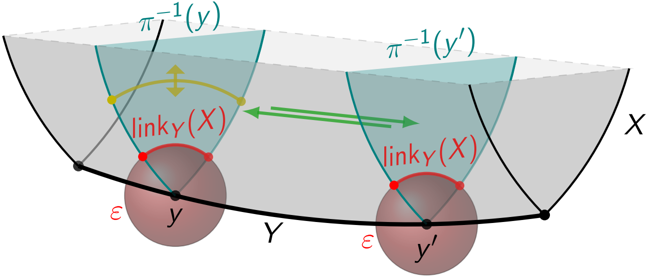
- We need to show that X looks like a closed half-space near any point on its boundary. The other properties follow by induction, using previous work.
- We use the framework of *links*, following Fomin and Shapiro (2000).



- It suffices to prove:
  1. $\mathrm{link}_Y(X)$ is homeomorphic to a closed ball; and
  2. locally near Y°, the space X looks like $Y^\circ \times \mathrm{cone}(\mathrm{link}_Y(X))$.
- We work instead with $\mathrm{link}_Y(X)$, where $Y \subset X$ is the cell containing y.

Fomin–Shapiro atlas

- To define $\mathrm{link}_Y(X)$, we need a projection $\pi : X \to Y$ and translations $\pi^{-1}(y') \to \pi^{-1}(y)$. To get (2), we need dilation actions on $\pi^{-1}(y)$.



- Fomin and Shapiro constructed the projections and translations for $U_n^{\geq 0}$ via matrix factorizations, using work of Kazhdan and Lusztig (1980).
- e.g. $\pi\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & \frac{ac-b}{a} \\ 0 & 0 & 1 \end{bmatrix} \leftsquigarrow \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & \frac{ac-b}{a} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & 1 \end{bmatrix}.$

1. How do we define the dilation actions?
2. How do we construct these maps for $\mathrm{Gr}_{k,n}^{\geq 0}$?

} *Snider's embedding* (2011)

Snider's embedding

- We fix I, and embed the subset of $\mathrm{Gr}_{k,n}$ where $\Delta_I \neq 0$ into the *affine flag variety* $\widetilde{\mathrm{Fl}}_n$, the set of n-periodic matrices modulo certain row operations.
- e.g. Let $I = \{1, 3\}$ with $k = 2, n = 4$. Then Snider's embedding is

$$\begin{bmatrix} 1 & a & 0 & b \\ 0 & c & 1 & d \end{bmatrix} \mapsto \begin{matrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & \\ \cdots & 0 & a & 0 & b & 1 & 0 & \cdots & & & & & \\ & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & \\ & & \cdots & 0 & d & 0 & c & 1 & 0 & \cdots & & & \\ & & & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & & \\ & & & & \cdots & 0 & a & 0 & b & 1 & 0 & \cdots & \\ & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \end{matrix} .$$

- We define the projection and translation maps by 'matrix factorizations' in $\widetilde{\mathrm{Fl}}_n$, which were studied by Knutson, Woo, and Yong (2013) for Fl_n.
- We obtain the dilation actions by translating to a 'hidden' point in $\widetilde{\mathrm{Fl}}_n$ in the closure of the image of Y (the analogue of a permutation matrix).
- For arbitrary G/P, we construct a generalization of Snider's embedding. A similar embedding was found independently by Huang (2019).

Open problems

- Show that the following spaces are regular CW complexes:
  1. Arkani-Hamed and Trnka's amplituhedra;
  2. Fomin and Zelevinsky's double Bruhat cells;
  3. totally nonnegative part of a Kac–Moody partial flag variety;
  4. Lam's compactified space of electrical networks;
  5. Galashin and Pylyavskyy's cell decomposition of the totally nonnegative orthogonal Grassmannian;
  6. Rietsch's totally nonnegative part of a Peterson variety;
  7. He's cell decomposition of pieces of the wonderful compactification.

Thank you!