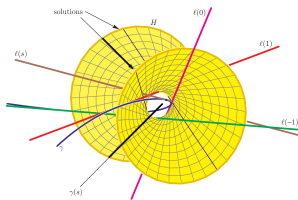


# Positivity in real Schubert calculus

Slides available at [snkarp.github.io](https://snkarp.github.io)



F. Sottile, "Frontiers of reality in Schubert calculus"

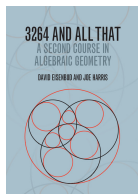


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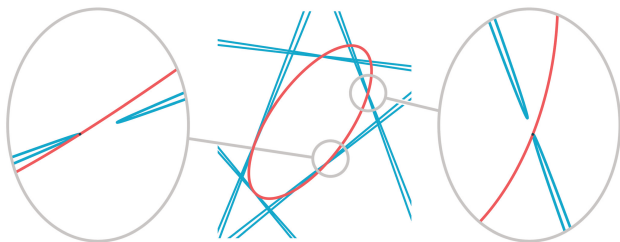
# Steiner's conic problem (1848)



- How many conics are tangent to 5 given conics? ~~7776~~.
- de Jonquières (1859): 3264.
- Fulton (1984): “The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems.”

- Fulton (1986); Ronga, Tognoli, Vust (1997): All 3264 conics can be real.

## 3264 Conics in a Second



- Breiding, Sturmfels, and Timme (2020) found 5 explicit such conics.

# The Grassmannian $\text{Gr}_{d,m}(\mathbb{C})$

- The Grassmannian  $\text{Gr}_{d,m}(\mathbb{C})$  is the set of  $d$ -dimensional subspaces of  $\mathbb{C}^m$ .

$$V := \begin{array}{c} \text{Diagram of a 2D plane in } \mathbb{C}^4 \text{ with origin } 0 \text{ and points } (1,0,-4,-3) \text{ and } (0,1,3,2) \end{array} = \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \text{Gr}_{2,4}(\mathbb{C})$$
$$= \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 6 & 4 \end{bmatrix}$$

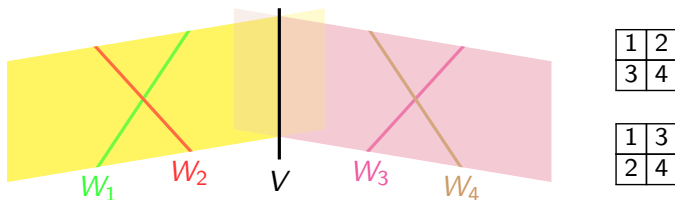
$$\Delta_{1,2} = 1, \quad \Delta_{1,3} = 3, \quad \Delta_{1,4} = 2, \quad \Delta_{2,3} = 4, \quad \Delta_{2,4} = 3, \quad \Delta_{3,4} = 1$$

$$\text{Plücker relation: } \Delta_{1,3}\Delta_{2,4} = \Delta_{1,2}\Delta_{3,4} + \Delta_{1,4}\Delta_{2,3}$$

- $\text{Gr}_{d,m}(\mathbb{C}) \cong \text{GL}_d(\mathbb{C}) \setminus \{d \times m \text{ matrices of rank } d\}$ .
- Given  $V \in \text{Gr}_{d,m}(\mathbb{C})$  as a  $d \times m$  matrix, for  $d$ -subsets  $I$  of  $\{1, \dots, m\}$  let  $\Delta_I(V)$  be the  $d \times d$  minor of  $V$  in columns  $I$ . The *Plücker coordinates*  $\Delta_I(V)$  are well-defined up to a common scalar.
- $\text{Gr}_{d,m}(\mathbb{C})$  is a projective variety of dimension  $d(m-d)$ .

# Schubert calculus (1886)

- Divisor Schubert problem: given  $W_1, \dots, W_{d(m-d)} \in \text{Gr}_{m-d, m}(\mathbb{C})$ , find all  $V \in \text{Gr}_{d, m}(\mathbb{C})$  such that  $V \cap W_i \neq \{0\}$  for all  $i$ .
- e.g.  $d = 2, m = 4$  (projectivized). Given 4 lines  $W_i \subseteq \mathbb{P}^3$ , find all lines  $V \subseteq \mathbb{P}^3$  intersecting all 4. Generically, there are 2 solutions.



We can see the 2 solutions explicitly when two pairs of the lines intersect.

- If the  $W_i$ 's are generic, the number of solutions  $V$  is

$$\#_{d,m} := \frac{1!2! \cdots (d-1)!}{(m-d)!(m-d+1)! \cdots (m-1)!} (d(m-d))!,$$

the number of *standard Young tableaux* of rectangular shape  $d \times (m-d)$ .

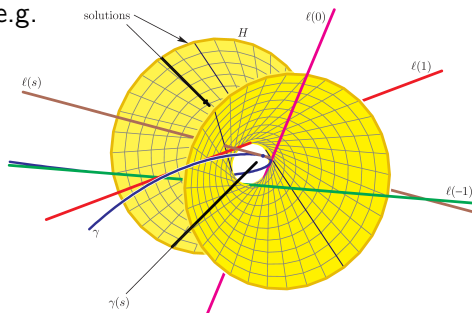
# Shapiro–Shapiro conjecture (1993)

- Are there Schubert problems with all solutions real?

## Shapiro–Shapiro conjecture (1993)

Let  $W_1, \dots, W_{d(m-d)} \in \text{Gr}_{m-d,m}(\mathbb{R})$  osculate the moment curve  $\gamma(t) := (1, t, \dots, t^{m-1})$  at real points. Then there are  $\#_{d,m}$  real solutions  $V \in \text{Gr}_{d,m}(\mathbb{R})$  such that  $V \cap W_i \neq \{0\}$  for all  $i$ .

- e.g.



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- The Schubert problem above arises in the study of linear series in algebraic geometry, differential equations, and pole placement problems in control theory.

- Bürgisser, Lerario (2020): a 'random' Schubert problem has  $\approx \sqrt{\#_{d,m}}$  real solutions.

# Shapiro–Shapiro conjecture (1993)

- Sottile (1999) tested the conjecture and proved it asymptotically.
- Eremenko and Gabrielov (2002) proved the conjecture for  $d = 2, m - 2$ .
- Mukhin, Tarasov, and Varchenko (2009) proved the full conjecture via the *Bethe ansatz*. The proof was simplified by Purbhoo (2022).
- Levinson and Purbhoo (2021) gave a topological proof of the conjecture.

## Secant conjecture, divisor form (Sottile (2003))

Let  $W_1, \dots, W_{d(m-d)} \in \text{Gr}_{m-d,m}(\mathbb{R})$  be secant to  $\gamma(t) := (1, t, \dots, t^{m-1})$  along non-overlapping intervals. Then there are  $\#_{d,m}$  real solutions

$$V \in \text{Gr}_{d,m}(\mathbb{R}) \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- The Shapiro–Shapiro conjecture is a limiting case of this conjecture.

## Theorem (Karp, Purbhoo (2023))

*The divisor form of the secant conjecture is true.*

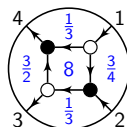
# Total positivity

- Total positivity originated in the 1910's from orthogonal polynomials. Gantmakher and Krein (1937) showed that *totally positive matrices* (whose minors are all positive) have positive eigenvalues.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \quad \begin{aligned} \lambda_1 &= 10.6031 \dots \\ \lambda_2 &= 1.2454 \dots \\ \lambda_3 &= 0.1514 \dots \end{aligned}$$

- Lusztig (1994) introduced total positivity for algebraic groups and flag varieties, motivated by quantum groups. An element  $V \in \text{Gr}_{d,m}(\mathbb{C})$  is *totally nonnegative* if its Plücker coordinates are all nonnegative.

$$V := \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \text{Gr}_{2,4}^{\geq 0}$$



$$\Delta_{1,2} = 1, \quad \Delta_{1,3} = 3, \quad \Delta_{1,4} = 2, \quad \Delta_{2,3} = 4, \quad \Delta_{2,4} = 3, \quad \Delta_{3,4} = 1$$

- Postnikov (2006) parametrized  $\text{Gr}_{d,m}^{\geq 0}$  using *plabic graphs*.
- $\text{Gr}_{d,m}^{\geq 0}$  appears in the study of cluster algebras, electrical networks, the KP hierarchy, scattering amplitudes, curve singularities, the Ising model, ...

# Wronski map

- The *Wronskian* of  $d$  functions  $f_1, \dots, f_d : \mathbb{C} \rightarrow \mathbb{C}$  is

$$\text{Wr}(f_1, \dots, f_d) := \det \begin{bmatrix} f_1 & \cdots & f_d \\ f_1' & \cdots & f_d' \\ \vdots & \ddots & \vdots \\ f_1^{(d-1)} & \cdots & f_d^{(d-1)} \end{bmatrix}.$$

- e.g.  $\text{Wr}(f, g) = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - f'g = f^2 \left(\frac{g}{f}\right)'$ .
- $\text{Wr}(f_1, \dots, f_d) \neq 0$  if and only if  $f_1, \dots, f_d$  are linearly independent. Then  $\text{Wr}(V)$  is well-defined up to a scalar, where  $V := \langle f_1, \dots, f_d \rangle$ .
- The monic linear differential operator  $\mathcal{L}$  of order  $d$  with kernel  $V$  is

$$\mathcal{L}(g) = \frac{\text{Wr}(f_1, \dots, f_d, g)}{\text{Wr}(f_1, \dots, f_d)} = g^{(d)} + \dots.$$

- We identify  $\mathbb{C}^m$  with the space of polynomials of degree at most  $m-1$ :

$$\mathbb{C}^m \leftrightarrow \mathbb{C}_{m-1}[u], \quad (a_1, \dots, a_m) \leftrightarrow a_1 + a_2 u + a_3 \frac{u^2}{2} + \cdots + a_m \frac{u^{m-1}}{(m-1)!}.$$

We obtain the *Wronski map*  $\text{Wr} : \text{Gr}_{d,m}(\mathbb{C}) \rightarrow \mathbb{P}(\mathbb{C}_{d(m-d)}[u])$ .



# Shapiro–Shapiro conjecture and positivity conjecture

## Shapiro–Shapiro conjecture (Mukhin, Tarasov, and Varchenko (2009))

Let  $V \in \text{Gr}_{d,m}(\mathbb{C})$ . If all complex zeros of  $\text{Wr}(V)$  are real, then  $V$  is real.

- e.g. If  $\text{Wr}(V) := (u + z_1)^2(u + z_2)^2$ , the two solutions  $V \in \text{Gr}_{2,4}(\mathbb{C})$  are

$$\begin{aligned} & \langle (u + z_1)(u + z_2), u(u + z_1)(u + z_2) \rangle & \text{and} & \quad \langle (u + z_1)^3, (u + z_2)^3 \rangle. \\ & = \begin{bmatrix} z_1 z_2 & z_1 + z_2 & 2 & 0 \\ 0 & z_1 z_2 & 2(z_1 + z_2) & 6 \end{bmatrix} & & = \begin{bmatrix} z_1^3 & 3z_1^2 & 6z_1 & 6 \\ z_2^3 & 3z_2^2 & 6z_2 & 6 \end{bmatrix} \end{aligned}$$

## Positivity conjecture (Mukhin, Tarasov (2017); Karp (2021))

Let  $V \in \text{Gr}_{d,m}(\mathbb{C})$ . If all zeros of  $\text{Wr}(V)$  are nonpositive, then  $V \in \text{Gr}_{d,m}^{\geq 0}$ .

- Karp (2023): The positivity conjecture is equivalent a conjecture of Eremenko (2015), which implies the divisor form of the secant conjecture.
- Karp, Purbhoo (2023): The positivity conjecture is true.

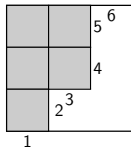
# Universal Plücker coordinates

- We want to find all  $V$  with  $\text{Wr}(V) = (u + z_1) \cdots (u + z_n)$ . It suffices to work in  $\text{Gr}_{n,2n}(\mathbb{C})$ . We construct *universal* Plücker coordinates  $\beta^\lambda \in \mathbb{C}[\mathfrak{S}_n]$  indexed by *partitions*  $\lambda$ .
- Partitions inside the  $n \times n$  square index  $n$ -element subsets of  $\{1, \dots, 2n\}$ .

$$\lambda = (2, 2, 1)$$

$$|\lambda| = 5$$

$$n = 3$$



$$\leftrightarrow I_\lambda = \{2, 4, 5\}$$

## Theorem (Karp, Purbhoo (2023))

- The  $\beta^\lambda$ 's pairwise commute and satisfy the Plücker relations.*
- There is a bijection between the eigenspaces of the  $\beta^\lambda$ 's acting on  $\mathbb{C}[\mathfrak{S}_n]$  and the elements  $V \in \text{Gr}_{n,2n}(\mathbb{C})$  with  $\text{Wr}(V) = (u + z_1) \cdots (u + z_n)$ , sending the eigenvalue of  $\beta^\lambda$  to the Plücker coordinate  $\Delta_{I_\lambda}(V)$ .*
- If  $z_1, \dots, z_n \geq 0$ , then the  $\beta^\lambda$ 's are positive semidefinite.*

# Definition of the universal Plücker coordinates

$$\beta^\lambda := \sum_{\substack{X \subseteq \{1, \dots, n\}, \\ |X| = |\lambda|}} \left( \prod_{i \notin X} z_i \right) \sum_{\pi \in \mathfrak{S}_X} \chi^\lambda(\pi) \pi \in \mathbb{C}[\mathfrak{S}_n]$$

• e.g.  $n = 2$ . Write  $\mathfrak{S}_2 = \{e, \sigma\}$ , where  $e$  is the identity. We have

$$\beta^\emptyset = z_1 z_2 e, \quad \beta^\square = (z_1 + z_2)e, \quad \beta^{\square\square} = e + \sigma, \quad \beta^{\square\square} = e - \sigma,$$

and  $\beta^\lambda = 0$  if  $|\lambda| > 2$ . On the eigenspace  $\langle e - \sigma \rangle$ , the eigenvalues are

$$\beta^\emptyset \rightsquigarrow z_1 z_2, \quad \beta^\square \rightsquigarrow z_1 + z_2, \quad \beta^{\square\square} \rightsquigarrow 0, \quad \beta^{\square\square} \rightsquigarrow 2,$$

which are the Plücker coordinates of

$$V = \begin{bmatrix} \frac{z_1+z_2}{2} & 1 & 0 & 0 \\ -z_1 z_2 & 0 & 2 & 0 \end{bmatrix} = \left\langle \frac{z_1 + z_2}{2} + u, -z_1 z_2 + u^2 \right\rangle \in \text{Gr}_{2,4}(\mathbb{C}).$$

We can check that

$$\text{Wr}(V) = \det \begin{bmatrix} \frac{z_1+z_2}{2} + u & -z_1 z_2 + u^2 \\ 1 & 2u \end{bmatrix} = (u + z_1)(u + z_2).$$

# KP hierarchy

- The *KP equation*

$$\frac{\partial}{\partial x} \left( -4 \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) + 3 \frac{\partial^2 u}{\partial y^2} = 0$$

models shallow water waves. It is the first equation in the *KP hierarchy*.



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- The solutions to the KP hierarchy are symmetric functions  $\tau(\mathbf{x})$  in the variables  $\mathbf{x} = (x_1, x_2, \dots)$  satisfying the *Hirota bilinear identity*:

$$[t^{-1}] \exp\left(\sum_{k \geq 1} \frac{t^k}{k} (p_k(\mathbf{x}) - p_k(\mathbf{y}))\right) \exp\left(\sum_{k \geq 1} -t^{-k} \left(\frac{\partial}{\partial p_k(\mathbf{x})} - \frac{\partial}{\partial p_k(\mathbf{y})}\right)\right) \tau(\mathbf{x}) \tau(\mathbf{y}) = 0.$$

- Sato (1981):  $\tau(\mathbf{x})$  satisfies the bilinear identity if and only if its coefficients in the Schur basis  $s_\lambda(\mathbf{x})$  satisfy the Plücker relations.
- A key to our proof is showing  $\sum_\lambda \beta^\lambda s_\lambda(\mathbf{x})$  satisfies the bilinear identity.

## Future directions

- Further explore the connection to the KP hierarchy.
- Find necessary and sufficient inequalities on the Plücker coordinates of  $V$  for all complex zeros of  $\text{Wr}(V)$  to be nonpositive, generalizing the Aissen–Schoenberg–Whitney theorem in the case  $\dim(V) = 1$ . (The positivity conjecture implies that the inequalities  $\Delta_I(V) \geq 0$  are necessary.)
- Address generalizations and variations of the Shapiro–Shapiro conjecture: the discriminant conjecture, the general form of the secant conjecture, the monotone conjecture, the total reality conjecture for convex curves, . . .

Thank you!