Gradient flows on totally nonnegative flag varieties

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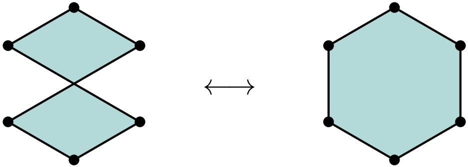

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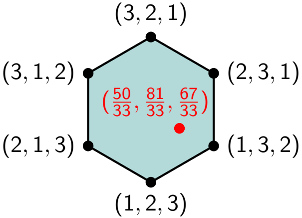
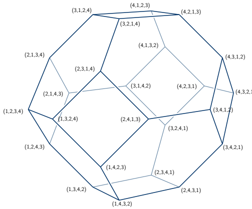
Schur–Horn theorem

- Let $\mathrm{Perm}(\lambda_1, \ldots, \lambda_n)$ be the polytope in $\mathbb{R}^n$ whose vertices are all permutations of $(\lambda_1, \ldots, \lambda_n)$.



$\mathrm{Perm}(3, 2, 1)$



$\mathrm{Perm}(4, 3, 2, 1)$

- Let μ send a matrix to its diagonal, e.g. $\mu\left(\frac{1}{33}\begin{bmatrix} 50 & 28 & 0 \\ 28 & 81 & 8 \\ 0 & 8 & 67 \end{bmatrix}\right) = \left(\frac{50}{33}, \frac{81}{33}, \frac{67}{33}\right)$.

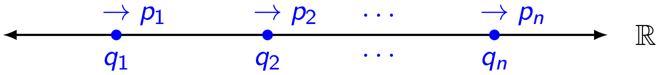
Theorem (Schur (1923), Horn (1953))

The map μ sends the space of $n \times n$ symmetric matrices with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ onto $\mathrm{Perm}(\lambda_1, \ldots, \lambda_n)$.

Toda lattice

- The *Toda lattice* (1967) is a Hamiltonian system with

$$H(\mathbf{q},\mathbf{p}) := \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} \qquad \left(\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i}\right).$$



- Flaschka (1974) expressed the Toda flow in *Lax form*: $\dot{L} = [L, \pi_{\text{skew}}(L)]$, where L is an $n \times n$ symmetric tridiagonal matrix with positive subdiagonal.

$$L = \begin{bmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \\ 0 & a_2 & b_3 \end{bmatrix}, \quad \pi_{\text{skew}}(L) = \begin{bmatrix} 0 & -a_1 & 0 \\ a_1 & 0 & -a_2 \\ 0 & a_2 & 0 \end{bmatrix}, \quad a_i = \tfrac{1}{2} e^{\frac{q_i - q_{i+1}}{2}}, \quad b_i = -\tfrac{1}{2} p_i.$$

- The eigenvalues of L are distinct and are invariant under the Toda flow. As $t \to \pm\infty$, L approaches a diagonal matrix with sorted diagonal entries.
- Let $\mathcal{J}_{\boldsymbol{\lambda}}^{>0}$ (respectively, $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$) denote the manifold of all L with fixed spectrum $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and all $a_i > 0$ (respectively, $a_i \geq 0$).

Isospectral manifold $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$

Theorem (Moser (1975))

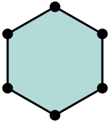
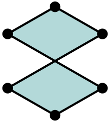
The map which sends $L \in \mathcal{J}_{\boldsymbol{\lambda}}^{>0}$ to the vector of first entries of its normalized eigenvectors is a homeomorphism onto $S_{>0}^{n-1}$.

- e.g. $L = \frac{1}{33}\begin{bmatrix} 50 & 28 & 0 \\ 28 & 81 & 8 \\ 0 & 8 & 67 \end{bmatrix} = \begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix}$

$$\mapsto \left(\tfrac{16}{33}, \tfrac{7}{33}, \tfrac{28}{33}\right) \in S_{>0}^{2}.$$

Theorem (Tomei (1984))

The space $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$ is homeomorphic to $\mathrm{Perm}(\boldsymbol{\lambda})$.

- However, $\mu : \mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0} \to \mathrm{Perm}(\boldsymbol{\lambda})$ is neither injective nor surjective.
- e.g. $\mathrm{Perm}(3,2,1) =$  , $\mu(\mathcal{J}_{(3,2,1)}^{\geq 0}) =$  .

Isospectral manifold of Jacobi matrices

Theorem (Bloch, Flaschka, Ratiu (1990))

Let Λ denote the diagonal matrix with diagonal $\boldsymbol{\lambda}$. Then the map

$$L = g\Lambda g^{-1} \mapsto \mu(g^{-1}\Lambda g) \qquad (g \in \mathrm{O}_n)$$

is a homeomorphism $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0} \to \mathrm{Perm}(\boldsymbol{\lambda})$, and is a diffeomorphism on $\mathcal{J}_{\boldsymbol{\lambda}}^{>0}$.

- e.g. $L = \begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix}$

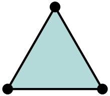
$$\mapsto \mu\left(\begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \right) = \left(\tfrac{795}{363}, \tfrac{401}{363}, \tfrac{982}{363}\right).$$

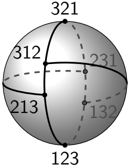
- The key to the proof is to define a map $L = g\Lambda g^{-1} \mapsto g^{-1}\Lambda g$ on $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$, by making a choice of $g \in \mathrm{O}_n$ which depends smoothly on $L \in \mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$. We show that total positivity provides a natural way to make this choice and to generalize it beyond the tridiagonal case.

Totally nonnegative flag varieties

- Let $K \subseteq \{1, \ldots, n-1\}$. The *partial flag variety* $\mathrm{Fl}_{K;n}(\mathbb{C})$ consists of tuples $V = (V_k)_{k \in K}$ of nested subspaces of $\mathbb{C}^n$, where $\dim(V_k) = k$.
- e.g. $\mathrm{Fl}_{\{1,3\};4}(\mathbb{C}) = \{(V_1, V_3) : V_1 \subset V_3 \subset \mathbb{C}^4, \dim(V_1) = 1, \dim(V_3) = 3\}$.
- Two special cases: when $K = \{1, \ldots, n-1\}$, we obtain the *complete flag variety* $\mathrm{Fl}_n(\mathbb{C})$; when $K = \{k\}$, we obtain the *Grassmannian* $\mathrm{Gr}_{k,n}(\mathbb{C})$.
- We say that $g \in \mathrm{GL}_n(\mathbb{C})$ *represents* $V \in \mathrm{Fl}_{K;n}(\mathbb{C})$ if each V_k is the span of the first k columns of g. We call V *totally positive* if it is represented by some g whose left-justified (i.e. initial) minors are all real and positive. We denote the set of such V by $\mathrm{Fl}_{K;n}^{>0}$. We similarly define $\mathrm{Fl}_{K;n}^{\geq 0}$.
- e.g. $\begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{4} & 1 & 0 \\ \frac{7}{16} & \frac{17}{4} & 1 \end{bmatrix} \in \mathrm{Fl}_3^{>0}$.

$\mathrm{Gr}_{1,3}^{\geq 0}$





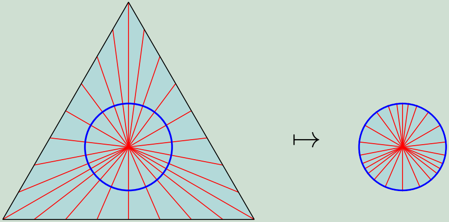
$\mathrm{Fl}_3^{\geq 0}$

Theorem (Galashin, Karp, Lam (2019))

The space $\mathrm{Fl}_{K;n}^{\geq 0}$ is homeomorphic to a closed ball.

Proof

Let M be the $n \times n$ tridiagonal matrix $\begin{bmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$. Then $V \mapsto \exp(tM)V$ for $t \in [0, \infty]$ contracts $\mathrm{Fl}_{K;n}^{\geq 0}$ onto a unique attractor in the interior.



Totally nonnegative adjoint orbits

- Let U_n be the group of $n \times n$ unitary matrices and $\mathfrak{u}_n$ its Lie algebra of $n \times n$ skew-Hermitian matrices. For $\lambda_1 \geq \cdots \geq \lambda_n$, consider the adjoint orbit

$$\mathcal{O}_{\boldsymbol{\lambda}} := \{g(\mathrm{i}\Lambda)g^{-1} : g \in \mathrm{U}_n\} \subseteq \mathfrak{u}_n, \qquad \text{where } \Lambda := \mathrm{Diag}(\lambda_1, \dots, \lambda_n).$$

- Let $K := \{1 \leq k \leq n-1 : \lambda_k > \lambda_{k+1}\}$. Then we have the isomorphism

$$\mathcal{O}_{\boldsymbol{\lambda}} \to \mathrm{Fl}_{K;n}(\mathbb{C}), \quad g(\mathrm{i}\Lambda)g^{-1} \mapsto g,$$

sending a matrix to its flag of eigenvectors ordered by descending eigenvalue.

- e.g. $\mathcal{O}_{(5,2,2,-1)} \cong \mathrm{Fl}_{\{1,3\};4}(\mathbb{C})$.
- We define $\mathcal{O}_{\boldsymbol{\lambda}}^{>0}$ and $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ to be the preimages of $\mathrm{Fl}_{K;n}^{>0}$ and $\mathrm{Fl}_{K;n}^{\geq 0}$.
- e.g. $\begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3\mathrm{i} & 0 & 0 \\ 0 & 2\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix} = \frac{\mathrm{i}}{33} \begin{bmatrix} 50 & 28 & 0 \\ 28 & 81 & 8 \\ 0 & 8 & 67 \end{bmatrix} \in \mathcal{O}_{(3,2,1)}^{>0}$.

Proposition (Bloch, Karp (2021))

If $\lambda_1 > \cdots > \lambda_n$, then the tridiagonal subset of $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ is precisely $\mathrm{i}\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$.

Gradient flows on adjoint orbits

- We consider the gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ of the function $L \mapsto 2n\,\mathrm{tr}(LN)$, where $N \in \mathfrak{u}_n$. We work in the *Kähler*, *normal*, and *induced* metrics. When $\mathcal{O}_{\boldsymbol{\lambda}} \cong \mathrm{Gr}_{k,n}(\mathbb{C})$, all three metrics coincide up to dilation.
- We say that the flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ *strictly preserves positivity* if trajectories starting in $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ lie in $\mathcal{O}_{\boldsymbol{\lambda}}^{>0}$ for all positive time. If so, we obtain a contractive flow with the Lyapunov function $L \mapsto -2n\,\mathrm{tr}(LN)$.

Proposition (Duistermaat, Kolk, Varadarajan (1983); Guest, Ohnita (1993))

The isomorphism $\mathcal{O}_{\boldsymbol{\lambda}} \cong \mathrm{Fl}_{K;n}(\mathbb{C})$ sends the gradient flow with respect to N in the Kähler metric to the flow $V(t) = \exp(t\mathrm{i}N)V$ on $\mathrm{Fl}_{K;n}(\mathbb{C})$.

- The contractive flow on $\mathrm{Fl}_{K;n}^{\geq 0}$ considered earlier is such a flow.

Theorem (Bloch, Karp (2021))

If $\mathcal{O}_{\boldsymbol{\lambda}} \ncong \mathrm{Gr}_{k,n}(\mathbb{C})$, then the gradient flow with respect to N in the Kähler metric strictly preserves positivity if and only if $\mathrm{i}N \in \mathcal{J}_{\boldsymbol{\mu}}^{>0}$ for some $\boldsymbol{\mu}$.

- We obtain a slightly larger family of N's when $\mathcal{O}_{\boldsymbol{\lambda}} \cong \mathrm{Gr}_{k,n}(\mathbb{C})$.

Gradient flows: normal and induced metrics

Proposition (Brockett (1991); Bloch, Brockett, Ratiu (1992))

The gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ with respect to N in the normal metric is

$$\dot{L} = [L, [L, N]].$$

Theorem (Bloch, Karp (2021))

If $\mathcal{O}_{\boldsymbol{\lambda}} \cong \mathrm{Fl}_n(\mathbb{C})$ with $n \geq 3$, then every gradient flow in the normal metric does not strictly preserve positivity.

Proposition (Bloch, Karp (2021))

The gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ with respect to N in the induced metric is

$$\dot{L} = [L, \mathrm{ad}_L^{-1}(N)].$$

Proposition (Bloch, Karp (2021))

Let $\lambda_1 > \lambda_2 > \lambda_3$ satisfy $\frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_3} \notin [\frac{1}{2+2\sqrt{2}}, 2 + 2\sqrt{2}]$. Then every gradient flow in the induced metric does not strictly preserve positivity.

Twist map

- Every element of $\mathrm{Fl}_n^{\geq 0}$ is represented by a unique $g \in \mathrm{U}_n$ whose left-justified minors are all nonnegative. Let $\vartheta(g) := ((-1)^{i+j}(g^{-1})_{i,j})_{1\leq i,j\leq n}$.
- e.g. $\vartheta\left(\frac{1}{33}\begin{bmatrix}16 & -7 & 28\\ 28 & -4 & -17\\ 7 & 32 & 4\end{bmatrix}\right) = \frac{1}{33}\begin{bmatrix}16 & -28 & 7\\ 7 & -4 & -32\\ 28 & 17 & 4\end{bmatrix} \overset{\mathrm{Fl}_n}{\equiv} \begin{bmatrix}16 & 16\cdot 3 & 16\cdot 3^2\\ 7 & 7\cdot 2 & 7\cdot 2^2\\ 28 & 28\cdot 1 & 28\cdot 1^2\end{bmatrix}$.

Theorem (Bloch, Karp (2021))

The involution ϑ defines a diffeomorphism $\mathrm{Fl}_n^{\geq 0} \to \mathrm{Fl}_n^{\geq 0}$.

- When $\mathcal{O}_{\boldsymbol{\lambda}} \cong \mathrm{Fl}_n(\mathbb{C})$, the map ϑ induces a map on $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$. Restricting to $\mathrm{i}\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$, we recover the map of Bloch, Flaschka, and Ratiu on $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$:

$$L = g\Lambda g^{-1} \mapsto g^{-1}\Lambda g, \qquad \text{where } \Lambda := \mathrm{Diag}(\lambda_1, \dots, \lambda_n).$$

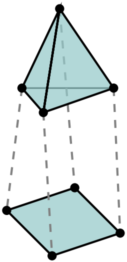
Proposition (Bloch, Karp (2021))

For $x \in \mathbb{R}^n_{>0}$, let $\mathrm{Vand}(\boldsymbol{\lambda}, x) \in \mathrm{Fl}_n(\mathbb{C})$ be the complete flag generated by $x, \Lambda x, \dots, \Lambda^{n-1}x$. Then the image of $\mathrm{i}\mathcal{J}_{\boldsymbol{\lambda}}^{>0} \subseteq \mathcal{O}_{\boldsymbol{\lambda}}^{>0} \cong \mathrm{Fl}_n^{>0}$ is

$$\vartheta(\{\mathrm{Vand}(\boldsymbol{\lambda}, x) : x \in \mathbb{R}^n_{>0}\}) \subseteq \mathrm{Fl}_n^{>0}.$$

Amplituhedra

- Let Z be a $(k+m)\times n$ matrix whose $(k+m)\times(k+m)$ minors are positive, which we regard as a linear map $Z : \mathbb{C}^n \to \mathbb{C}^{k+m}$. The *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is the image of the induced map $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}(\mathbb{C})$.



$$\begin{array}{c} \mathrm{Gr}_{k,n}^{\geq 0} \\ \tilde{Z} \Big\downarrow \; \text{linear map } \mathbb{C}^n \to \mathbb{C}^{k+m} \\ \textit{amplituhedron} \\ \mathcal{A}_{n,k,m}(Z) \subseteq \mathrm{Gr}_{k,k+m}(\mathbb{C}) \end{array}$$

- When $m = 4$, $\mathcal{A}_{n,k,m}(Z)$ encodes the tree-level scattering amplitude in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory.
- It is expected that $\mathcal{A}_{n,k,m}(Z)$ is homeomorphic to a closed ball. This is known for $k + m = n$; $k = 1$; $m = 1$; $n - k - m = 1$ with m even; and the family of *cyclically symmetric amplituhedra*.

Gradient flows on amplituhedra

Proposition (Bloch, Karp (2021))

Let $Z : \mathbb{C}^n \to \mathbb{C}^{k+m}$. Then $\tilde{Z} : \mathrm{Gr}_{k,n}(\mathbb{C}) \dashrightarrow \mathrm{Gr}_{k,k+m}(\mathbb{C})$ coherently projects gradient flows with respect to $N \in \mathfrak{u}_n$ if and only if $\ker(Z)$ is N-invariant.

- If the gradient flow on $\mathrm{Gr}_{k,n}(\mathbb{C})$ with respect to N strictly preserves positivity, then we obtain a contractive flow on $\mathcal{A}_{n,k,m}(Z)$.

Theorem (Bloch, Karp (2021))

Let Z be any $(k+m) \times n$ matrix whose rows form a basis for the $(k+m)$-dimensional subspace of the twisted Vandermonde flag

$$\vartheta(\mathrm{Vand}(\boldsymbol{\lambda}, x)) \quad (\lambda_1 > \cdots > \lambda_n,\ x \in \mathbb{R}^n_{>0}).$$

Then $\mathcal{A}_{n,k,m}(Z)$ is homeomorphic to a closed ball.

- e.g. $\vartheta\left(\begin{bmatrix} 16 & 16 \cdot 3 & 16 \cdot 3^2 \\ 7 & 7 \cdot 2 & 7 \cdot 2^2 \\ 28 & 28 \cdot 1 & 28 \cdot 1^2 \end{bmatrix}\right) = \frac{1}{33}\begin{bmatrix} 16 & -7 & 28 \\ 28 & -4 & -17 \\ 7 & 32 & 4 \end{bmatrix} \rightsquigarrow Z = \frac{1}{33}\begin{bmatrix} 16 & 28 & 7 \\ -7 & -4 & 32 \end{bmatrix}.$
- In particular, every amplituhedron with $n - k - m \le 2$ is a closed ball.

Future directions

- Extend the setup to more general adjoint orbits beyond the case of $\mathfrak{u}_n$.
- Classify positivity-preserving flows in the normal and induced metrics.
- Study flows on the cell closures of the cell decomposition of $\mathrm{Fl}_{K;n}^{\geq 0}$.
- Study the subset of $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ with bounded bandwidth (the tridiagonal subset being $\mathrm{i}\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$).
- Classify gradient flows preserving amplituhedra (which do not necessarily come from projections).
- Do twisted Vandermonde amplituhedra have any other special properties?

Thank you!