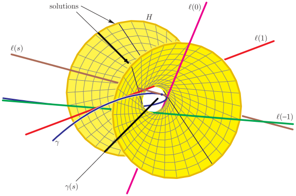
Positivity in real Schubert calculus

Slides available at `snkarp.github.io`



F. Sottile, "Frontiers of reality in Schubert calculus"

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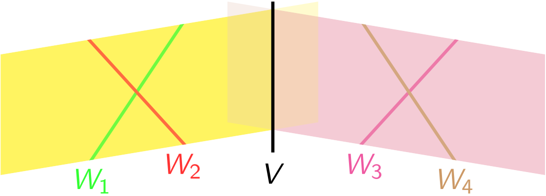
Schubert calculus (1886)

- Divisor Schubert problem: given subspaces $W_1, \ldots, W_{d(m-d)} \subseteq \mathbb{C}^m$ of dimension $m-d$, find all

$$d\text{-subspaces } V \subseteq \mathbb{C}^m \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- e.g. $d=2$, $m=4$ (projectivized). Given 4 lines $W_i \subseteq \mathbb{CP}^3$, find all lines $V \subseteq \mathbb{CP}^3$ intersecting all 4. Generically, there are 2 solutions.



1	2
3	4

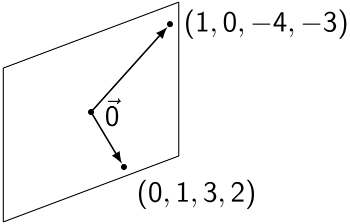
1	3
2	4

We can see the 2 solutions explicitly when two pairs of the lines intersect.

- If the W_i's are generic, the number of solutions V is $f^{\square}$, the number of *standard Young tableaux* of rectangular shape $d \times (m-d)$.
- Fulton (1984): "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems."

The Grassmannian $\mathrm{Gr}_{d,m}(\mathbb{C})$

- The *Grassmannian* $\mathrm{Gr}_{d,m}(\mathbb{C})$ is the set of d-dimensional subspaces of $\mathbb{C}^m$.

$V :=$ 

$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 6 & 4 \end{bmatrix} \in \mathrm{Gr}_{2,4}(\mathbb{C})$$

$$\Delta_{1,2} = 1, \ \ \Delta_{1,3} = 3, \ \ \Delta_{1,4} = 2, \ \ \Delta_{2,3} = 4, \ \ \Delta_{2,4} = 3, \ \ \Delta_{3,4} = 1$$

Plücker relation: $\Delta_{1,3}\Delta_{2,4} = \Delta_{1,2}\Delta_{3,4} + \Delta_{1,4}\Delta_{2,3}$

- Given $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ as a $d \times m$ matrix, for d-subsets J of $\{1, \ldots, m\}$ let $\Delta_J(V)$ be the $d \times d$ minor of V in columns J. The *Plücker coordinates* $\Delta_J(V)$ are well-defined up to a common scalar.
- $\mathrm{Gr}_{d,m}(\mathbb{C})$ is a projective variety of dimension $d(m-d)$.

Shapiro–Shapiro conjecture

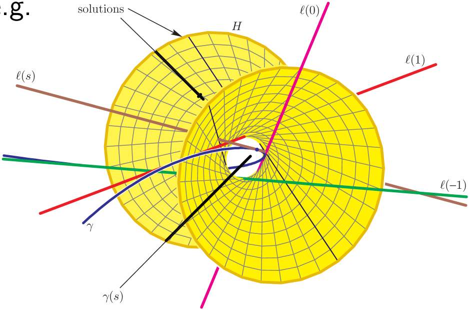
- Do there exist Schubert problems with all real solutions?

Shapiro–Shapiro conjecture (1993)

Let $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ *osculate the moment curve* $\gamma(t) := (\frac{t^{m-1}}{(m-1)!}, \frac{t^{m-2}}{(m-2)!}, \ldots, t, 1)$ *at real points. Then there exist* $f^{\square}$

real $V \in \mathrm{Gr}_{d,m}(\mathbb{R})$ *such that* $V \cap W_i \neq \{0\}$ *for all* i.

- e.g.



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- This Schubert problem arises in the study of linear series in algebraic geometry, differential equations, and pole placement problems in control theory.
- Bürgisser–Lerario (2020): a uniformly random Schubert problem over $\mathbb{R}$ has $\approx\sqrt{f^{\square}}$ real solutions.

Shapiro–Shapiro conjecture and secant conjecture

- Sottile (1999) tested the conjecture and proved it asymptotically.
- Eremenko–Gabrielov (2002): cases $d \leq 2$, $m - d \leq 2$.
- Mukhin–Tarasov–Varchenko (2009): full conjecture via the *Bethe ansatz*.
- Levinson–Purbhoo (2021): topological proof of the full conjecture.
- Vakil (2006): reality of Grassmannian Schubert calculus.

Secant conjecture, divisor form (Sottile (2003))

Let $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ be secant to the moment curve $\gamma(t)$ along non-overlapping real intervals. Then there exist $f^{\square}$

$$\textit{real } V \in \mathrm{Gr}_{d,m}(\mathbb{R}) \textit{ such that } V \cap W_i \neq \{0\} \textit{ for all } i.$$

- Eremenko–Gabrielov–Shapiro–Vainshtein (2006): case $m - d \leq 2$.

Theorem (Karp–Purbhoo (2023))

The divisor form of the secant conjecture is true.

Total positivity

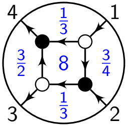
- *Totally positive matrices* (matrices whose minors are all positive) have been studied since the 1930's. Gantmakher–Krein (1937) showed that square totally positive matrices have positive eigenvalues.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 10.6031\cdots \\ \lambda_2 &= 1.2454\cdots \\ \lambda_3 &= 0.1514\cdots \end{aligned}$$

- Lusztig (1994) introduced total positivity for algebraic groups G and flag varieties G/P. An element $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ is *totally nonnegative* if its Plücker coordinates are all nonnegative.

$$V := \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

$$\Delta_{1,2} = 1,\ \Delta_{1,3} = 3,\ \Delta_{1,4} = 2,\ \Delta_{2,3} = 4,\ \Delta_{2,4} = 3,\ \Delta_{3,4} = 1$$



- Postnikov (2006) parametrized $\mathrm{Gr}_{d,m}^{\geq 0}$ using *plabic graphs*.
- $\mathrm{Gr}_{d,m}^{\geq 0}$ is related to cluster algebras, electrical networks, the KP hierarchy, scattering amplitudes, curve singularities, the Ising model, knot theory, ...

Positive Shapiro–Shapiro conjecture

Positivity conjecture (Mukhin–Tarasov (2017), Karp (2021))

Let $W_1, \dots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ osculate the moment curve $\gamma(t)$ at real points $t_1, \dots, t_{d(m-d)} \geq 0$. Then there exist $f^{\square}$

$$\textit{totally nonnegative } V \in \mathrm{Gr}_{d,m}^{\geq 0} \textit{ such that } V \cap W_i \neq \{0\} \textit{ for all } i.$$

- e.g. $d = 2$, $m = 4$. If $t_3, t_4 \to \infty$, then the 2 solutions $V \in \mathrm{Gr}_{2,4}(\mathbb{C})$ are

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t_1 t_2 & t_1 + t_2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{t_1+t_2}{2} & 1 & 0 & 0 \\ -t_1 t_2 & 0 & 2 & 0 \end{bmatrix}.$$

- Karp (2023): the positivity conjecture is equivalent to a conjecture of Eremenko (2015), which implies the divisor form of the secant conjecture.

Theorem (Karp–Purbhoo (2023))

The positivity conjecture is true.

- To prove it, we explicitly solve for the $\Delta_J(V)$'s over $\mathbb{C}[\mathfrak{S}_{d(m-d)}]$.

Universal Plücker coordinates

- Shapiro–Shapiro problem: given $W_1, \dots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{C})$ which osculate the moment curve $\gamma(t)$ at $t_1, \dots, t_{d(m-d)} \in \mathbb{C}$, find all

$$V \in \mathrm{Gr}_{d,m}(\mathbb{C}) \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

Theorem (Karp–Purbhoo (2023))

There exist linear operators $\beta_J = \beta_J(t_1, \dots, t_{d(m-d)})$ indexed by d-subsets $J \subseteq \{1, \dots, m\}$ with the following properties.

(i) The β_J's commute and satisfy the Plücker relations.

(ii) There is a bijection between the common eigenspaces of the β_J's and the solutions V above, sending the eigenvalue of β_J to $\Delta_J(V)$.

(iii) If $t_1, \dots, t_{d(m-d)} \geq 0$, then the β_J's are positive semidefinite.

$$\beta_J := \sum_{\substack{X \subseteq \{1,\dots,n\}, \\ |X| = |\lambda(J)|}} \Big(\prod_{i \notin X} t_i\Big) \sum_{\pi \in \mathfrak{S}_X} \chi^{\lambda(J)}(\pi)\pi \in \mathbb{C}[\mathfrak{S}_n] \quad (n = d(m-d))$$

Example: $d = 2$, $m = 4$, and $t_3, t_4 \to \infty$

$$\beta_J := \sum_{\substack{X \subseteq \{1,\dots,n\},\\ |X| = |\lambda(J)|}} \Big(\prod_{i \notin X} t_i\Big) \sum_{\pi \in \mathfrak{S}_X} \chi^{\lambda(J)}(\pi)\pi \in \mathbb{C}[\mathfrak{S}_n] \quad (n = 2)$$

- Write $\mathfrak{S}_2 = \{e, \sigma\}$, where e is the identity and $\sigma = (1\ \ 2)$. We have

$$\beta_{1,2} \overset{\varnothing}{=} t_1 t_2 e, \qquad \beta_{1,3} \overset{\square}{=} (t_1 + t_2)e, \qquad \beta_{1,4} \overset{\square\square}{=} e + \sigma, \qquad \beta_{2,3} \overset{\substack{\square\\ \square}}{=} e - \sigma,$$

and $\beta_J = 0$ otherwise. The β_J's commute and satisfy the Plücker relation

$$\beta_{1,3}\beta_{2,4} = \beta_{1,2}\beta_{3,4} + \beta_{1,4}\beta_{2,3} \quad \rightsquigarrow \quad 0 = 0 + (e + \sigma)(e - \sigma).$$

- On the eigenspace $\langle e - \sigma \rangle$, the eigenvalues are

$$\beta_{1,2} \rightsquigarrow t_1 t_2, \qquad \beta_{1,3} \rightsquigarrow t_1 + t_2, \qquad \beta_{1,4} \rightsquigarrow 0, \qquad \beta_{2,3} \rightsquigarrow 2,$$

which are the Plücker coordinates of

$$V = \begin{bmatrix} \frac{t_1 + t_2}{2} & 1 & 0 & 0 \\ -t_1 t_2 & 0 & 2 & 0 \end{bmatrix} \in \mathrm{Gr}_{2,4}(\mathbb{C}).$$

Proof 1: KP hierarchy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- The key to the proof is showing that the β_J's satisfy the Plücker relations.
- The KP equation models shallow waves. It is the first equation in the *KP hierarchy*, whose solutions are symmetric functions $\tau(\mathbf{x})$ in $\mathbf{x} = (x_1, x_2, \dots)$ satisfying *Hirota's identity*

$$[t^{-1}]\big(B_{\mathbf{x}}(t)\tau(\mathbf{x}) \cdot B_{\mathbf{y}}^{\perp}(t^{-1})\tau(\mathbf{y})\big) = 0.$$

Here $\cdot^{\perp}$ denotes the adjoint with respect to $\langle \cdot, \cdot \rangle$ (so $p_k(\mathbf{x})^{\perp} = k\frac{\partial}{\partial p_k(\mathbf{x})}$), and

$$B_{\mathbf{x}}(t) := H_{\mathbf{x}}(t)E_{\mathbf{x}}^{\perp}(-t^{-1}), \quad H_{\mathbf{x}}(t) := \sum_{k \geq 0} h_k(\mathbf{x})t^k, \quad E_{\mathbf{x}}(t) := \sum_{k \geq 0} e_k(\mathbf{x})t^k.$$

- Sato (1981): $\tau(\mathbf{x})$ satisfies Hirota's identity if and only if its coefficients in the Schur basis $s_\lambda(\mathbf{x})$ satisfy the Plücker relations.
- Karp–Purbhoo (2023): $\sum_J \beta_J s_{\lambda(J)}(\mathbf{x})$ satisfies Hirota's identity.

Proof 2: higher Gaudin Hamiltonians

- The *higher Gaudin Hamiltonian* associated to the partition λ is

$$T_\lambda := (t_1 + \mathbf{d}_1) \cdots (t_n + \mathbf{d}_n) s_\lambda(h) \in \mathrm{End}\big((\mathbb{C}^d)^{\otimes n}\big),$$

where:

  - h is a $d \times d$ matrix;
  - $s_\lambda(h)$ is the Schur polynomial evaluated at the eigenvalues of h; and
  - $\mathbf{d}_i$ is the derivative with respect to h^T acting in the ith tensor factor.

Theorem (Alexandrov–Leurent–Tsuboi–Zabrodin (2014))

The T_λ's pairwise commute and satisfy the Plücker relations.

Theorem (Karp–Mukhin–Tarasov (2024))

(i) We have $\beta_J = T_{\lambda(J)}|_{h=0}$.

(ii) If $t_1, \ldots, t_n \geq 0$ and h is positive semidefinite, then so is T_λ.

- Part (ii) gives a positivity theorem for spaces of *quasi-exponentials*.

Computing with higher Gaudin Hamiltonians

- e.g. $d=2$, $n=2$. Let us verify that $T_{\square\square}|_{h=0} = \beta_{1,4}$, i.e.,
$$\mathbf{d}_2\,\mathbf{d}_1\,s_{\square\square}(h) = e + \sigma \in \mathrm{End}\big((\mathbb{C}^2)^{\otimes 2}\big), \qquad \text{where } \mathfrak{S}_2 = \{e,\sigma\}.$$
- Denote $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so that $\mathbf{d}\,\phi(h) = \begin{bmatrix} \partial_a\phi & \partial_c\phi \\ \partial_b\phi & \partial_d\phi \end{bmatrix}$. We have
$$s_{\square\square}(h) = \frac{p_{\substack{\square\\\square}}(h) + p_{\square\square}(h)}{2} = \frac{\mathrm{Tr}(h)^2 + \mathrm{Tr}(h^2)}{2} = a^2 + d^2 + ad + bc.$$
- Then
$$\mathbf{d}_1\,s_{\square\square}(h) = \begin{bmatrix} 2a+d & b \\ c & a+2d \end{bmatrix},$$
$$\mathbf{d}_2\,\mathbf{d}_1\,s_{\square\square}(h) = \mathbf{d}_2\left((2a+d)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (a+2d)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$
$$= (v \otimes w \mapsto v \otimes w + w \otimes v) = e + \sigma.$$

Future directions

- Further explore the connection to the KP hierarchy.
- What happens to the higher Gaudin Hamiltonian T_λ if s_λ is replaced by a different symmetric function?
- Address generalizations and variations of the Shapiro–Shapiro conjecture: the discriminant conjecture, the general form of the secant conjecture, the monotone conjecture, the total reality conjecture for convex curves, ...

Thank you!