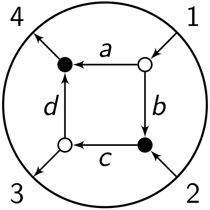
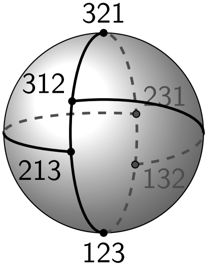
Introduction to total positivity

Slides available at `snkarp.github.io`





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Pre-history

- Let $\mathrm{var}(v)$ denote the number of sign changes of $v \in \mathbb{R}^n$.

$$\mathrm{var}(0, 4, 3, 0, -1, -3, 0, 7, 5) = 2$$

Theorem (Descartes (1637))

The nonzero real polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in \mathbb{R}[x]$ has at most $\mathrm{var}(a_0, \ldots, a_n)$ positive real zeros.

Theorem (Perron (1907), Frobenius (1912))

Let A be an $n \times n$ matrix with positive entries. Then A has a simple eigenvalue $\lambda > 0$ such that $\lambda > |\mu|$ for all other eigenvalues $\mu \in \mathbb{C}$. The eigenvector $v \in \mathbb{R}^n$ of λ has positive entries (up to rescaling).

- e.g.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \lambda = 16.12\cdots \qquad v = \begin{bmatrix} 0.232\cdots \\ 0.525\cdots \\ 0.819\cdots \end{bmatrix}$$

Totally positive kernels

- A *kernel* is a continuous function $K : [0,1]^2 \to \mathbb{R}$. It acts on $\mathcal{C}([0,1])$:
$$(Kf)(x) := \int_0^1 K(x,y)f(y)dy.$$
- We call K *totally positive* if
$$\det \begin{bmatrix} K(x_1,y_1) & \cdots & K(x_1,y_n) \\ \vdots & \ddots & \vdots \\ K(x_n,y_1) & \cdots & K(x_n,y_n) \end{bmatrix} > 0 \qquad \begin{array}{c} \text{for all } n \geq 1, \\ 0 \leq x_1 < \cdots < x_n \leq 1, \\ 0 \leq y_1 < \cdots < y_n \leq 1. \end{array}$$
- e.g. $K(x,y) = e^{xy}$.

Theorem (Kellogg (1918), Gantmakher (1936))

Let $K : [0,1]^2 \to \mathbb{R}$ be a totally positive kernel.
(i) The eigenvalues of K are positive and distinct: $\lambda_1 > \lambda_2 > \cdots > 0$.
(ii) If the corresponding eigenfunctions are $f_1, f_2, \ldots,$ then f_k has exactly $k-1$ zeros on $[0,1]$, and the zeros of f_k and f_{k+1} on $[0,1]$ interlace.

Totally positive matrices

- A matrix is *totally positive* if all of its minors are positive.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \quad \begin{aligned} \lambda_1 &= 10.60\cdots \\ \lambda_2 &= 1.24\cdots \\ \lambda_3 &= 0.15\cdots \end{aligned} \quad v_1 = \begin{bmatrix} 0.13\cdots \\ 0.43\cdots \\ 0.89\cdots \end{bmatrix} \quad v_2 = \begin{bmatrix} 0.81\cdots \\ 0.50\cdots \\ -0.30\cdots \end{bmatrix} \quad v_3 = \begin{bmatrix} 0.66\cdots \\ -0.73\cdots \\ 0.17\cdots \end{bmatrix}$$

minor $= 3$

Theorem (Gantmakher–Krein (1937, 1950))

Let A be an $n \times n$ totally positive matrix.

(i) The eigenvalues of A are positive and distinct: $\lambda_1 > \cdots > \lambda_n > 0$.

(ii) If the corresponding eigenvectors are $v_1, \ldots, v_n$, then $\mathrm{var}(v_k) = k - 1$, and the sign changes of v_k and v_{k+1} interlace.

- Proof: apply the Perron–Frobenius theorem to A acting on $\bigwedge^k \mathbb{R}^n$.
- Koteljanskiĭ (1963): For part (i), we only need the principal and almost-principal minors of A to be positive.

Work of Schoenberg

- Pólya (1912): Which linear maps $A : \mathbb{R}^m \to \mathbb{R}^n$ weakly diminish sign variation (i.e. $\mathrm{var}(Av) \leq \mathrm{var}(v)$ for all $v \in \mathbb{R}^m$)?
- Motivation: a discrete analogue of functions which weakly decrease the number of real zeros of a continuous function.

Theorem (Schoenberg (1930))

If A is injective, then A weakly diminishes sign variation if and only if for all $1 \leq k \leq m$, all nonzero $k \times k$ minors of A have the same sign.

- e.g.

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

- Motzkin (1933) solved the problem for all A.

Real-rooted polynomials

Theorem (Aissen–Schoenberg–Whitney (1952))

Let $f(x) := a_0 + a_1 x + \cdots + a_n x^n$ have at least one positive coefficient. Then all complex zeros of f are ≤ 0 if and only if the Toeplitz matrix

$$M := \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is totally nonnegative (i.e. all its minors are ≥ 0).

- Edrei (1952) extended the result to power series. It can be used to state the Riemann hypothesis: M is totally nonnegative when $f(x) := \xi(\sqrt{x} + \frac{1}{2})$.
- If all 2×2 minors of M are nonnegative, then $(a_0, \ldots, a_n)$ is log-concave.

Theorem (Schoenberg (1955))

Fix $k \geq 1$. If all $k \times k$ minors of M are nonnegative, then f has no complex zeros $z \neq 0$ in the sector $|\arg(z)| < \frac{k}{n+k-1}\pi$, and this bound is tight.

Tests and parametrizations

- Adjacent positive row and column operations on $m \times n$ matrices preserve total nonnegativity. Whitney (1952) inverted these operations to test for total nonnegativity in time $O((m+n)^3)$ (also called "Neville elimination").

Corollary (Loewner (1955))

For $1 \le i \le n-1$, define matrices differing from the identity I_n as follows:

$$x_i(t) := \begin{array}{c} \\ {\scriptstyle i} \\ {\scriptstyle i+1} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle i} & {\scriptstyle i+1} \end{array} \\ \left[\begin{array}{cc} 1 & t \\ 0 & 1 \end{array}\right] \end{array} \quad \text{and} \quad y_i(t) := x_i(t)^{\mathsf{T}} = \begin{array}{c} \\ {\scriptstyle i} \\ {\scriptstyle i+1} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle i} & {\scriptstyle i+1} \end{array} \\ \left[\begin{array}{cc} 1 & 0 \\ t & 1 \end{array}\right] \end{array}.$$

Let $(i_1, \dots, i_\ell)$ and $(j_1, \dots, j_\ell)$ be reduced words for $w_0 \in \mathfrak{S}_n$, where $\ell := \binom{n}{2}$. Then every totally positive $n \times n$ matrix is uniquely expressed as

$$y_{j_\ell}(*) \cdots y_{j_1}(*) \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} x_{i_1}(*) \cdots x_{i_\ell}(*), \qquad \text{where the $*$'s are in } \mathbb{R}_{>0}.$$

- e.g. $n = 3$. $\begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$

Lusztig's theory of total positivity for G (1994)

- Let G be a real reductive algebraic group with simple roots I. Fix a *pinning* $(T, B, B_-, x_i, y_i)_{i \in I}$. Then:
  - $G_{\geq 0}$ is the semigroup generated by $x_i(t)$, $y_i(t)$, and $T_{>0}$ for $t > 0$; and
  - $G_{>0} := y_{j_\ell}(*) \cdots y_{j_1}(*) \cdot T_{>0} \cdot x_{i_1}(*) \cdots x_{i_\ell}(*)$, where the $*$'s are in $\mathbb{R}_{>0}$, and $(i_1, \dots, i_\ell)$ and $(j_1, \dots, j_\ell)$ are reduced words for the longest element w_0 of the Weyl group W.

Theorem (Lusztig (1994))

Every $g \in G_{>0}$ is conjugate to an element of $T_{>0}$.

- Proof: apply the Perron–Frobenius theorem to g acting on the irreducible G-module V_λ, which has positive coefficients in the *canonical basis* of V_λ defined by Lusztig (1990).
- Fomin–Zelevinsky (2000) characterized $G_{\geq 0}$ and $G_{>0}$ by nonnegativity and positivity, respectively, of *generalized minors*.

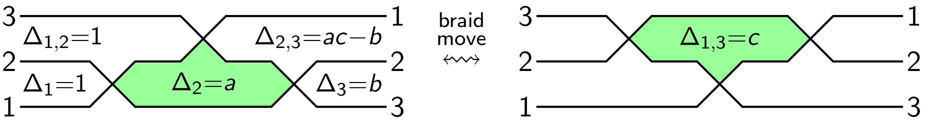
Fomin–Zelevinsky's cluster algebras (2000)

- Let U_n be the set of upper-triangular unipotent $n \times n$ matrices, and $U_n^{>0}$ be the subset where all minors (which are not identically zero) are positive.
- e.g. $U_3^{>0} = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, \cancel{c}, ac - b > 0 \right\}$

Theorem (Cryer (1972); Gasca–Peña (1992))

$U_n^{>0}$ is the subset of U_n where all contiguous topmost minors are positive.

- Berenstein–Fomin–Zelevinsky's proof (1996):



(positive) exchange relation: $\Delta_2\Delta_{1,3} = \Delta_1\Delta_{2,3} + \Delta_{1,2}\Delta_3$

- For larger n, we get exchange relations not coming from braid moves. These generate positive functions on $U_n^{>0}$ called *cluster variables*.

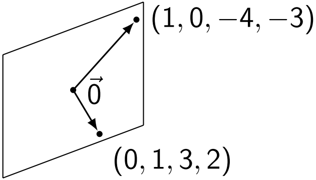
Totally nonnegative partial flag varieties $(G/P)_{\geq 0}$

- Lusztig (1998): Given a parabolic subgroup $P \subseteq G$, define

$$(G/P)_{>0} := G_{>0}/P \quad \text{and} \quad (G/P)_{\geq 0} := \overline{(G/P)_{>0}}.$$

- e.g. Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{C}) := \{k\text{-dimensional subspaces of } \mathbb{C}^n\}$.

$V :=$



$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{>0}$$

$$\Delta_{1,2} = 1, \;\; \Delta_{1,3} = 3, \;\; \Delta_{1,4} = 2, \;\; \Delta_{2,3} = 4, \;\; \Delta_{2,4} = 3, \;\; \Delta_{3,4} = 1$$

- The *Plücker coordinates* $\Delta_I(V)$ are the $k \times k$ minors of V (modulo global rescaling), indexed by k-subsets I of $\{1, \ldots, n\}$.
- Rietsch (2009): $\mathrm{Gr}_{k,n}^{>0} = \{V \in \mathrm{Gr}_{k,n}(\mathbb{C}) : \Delta_I(V) > 0 \text{ for all } I\}$ (and similarly for $\mathrm{Gr}_{k,n}^{\geq 0}$).

Totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$

Theorem (Gantmakher–Krein (1950))

Let $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$*.*
(i) We have $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ *if and only if* $\mathrm{var}(v) \leq k-1$ *for all* $v \in V$*.*
(ii) We have $V \in \mathrm{Gr}_{k,n}^{>0}$ *if and only if* $\mathrm{var}(w) \geq k$ *for all nonzero* $w \in V^{\perp}$*.*

- We have $\{m \times n \text{ totally positive matrices}\} \cong \mathrm{Gr}_{m,m+n}^{>0}$.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \hookrightarrow \begin{bmatrix} 1 & 0 & -d & -e & -f \\ 0 & 1 & a & b & c \end{bmatrix} \in \mathrm{Gr}_{2,5}(\mathbb{C})$$

We can regard $\mathrm{Gr}_{m,m+n}^{\geq 0}$ as a natural compactification.

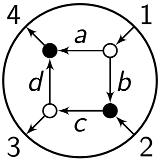
Theorem (Purbhoo (2018))

If $V \in \mathrm{Gr}_{k,n}^{\geq 0}$*, then the polynomial* $\sum_I \Delta_I(V) \cdot \prod_{i \in I} x_i$ *is stable (i.e. it is nonzero on* $\mathcal{H}^n$*, where* $\mathcal{H} \subseteq \mathbb{C}$ *is the upper half-plane).*

Cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

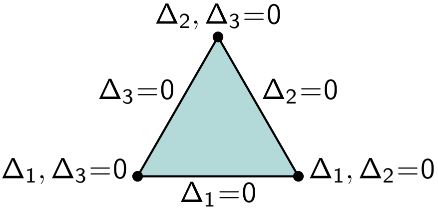
- Postnikov (2006): $\mathrm{Gr}_{k,n}^{\geq 0}$ decomposes into *positroid cells*, by specifying whether each Δ_I is zero or positive. Every cell is parametrized by a *plabic graph*.

$$\mathrm{Gr}_{2,4}^{>0} = \begin{bmatrix} 1 & 0 & -bc & -a-bcd \\ 0 & 1 & c & cd \end{bmatrix} \quad \leftrightsquigarrow$$



- $\mathrm{Gr}_{1,n}^{\geq 0}$ is the standard simplex in $\mathbb{RP}^{n-1}$. Lam (2014) studied *Grassmann polytopes*, motivated by scattering amplitudes. This led to the theory of *positive geometries*.

$$\mathrm{Gr}_{1,3}^{\geq 0} = \mathbb{P}_{\geq 0}^{2} \cong$$

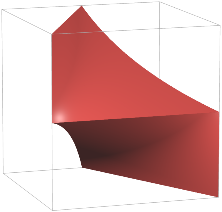


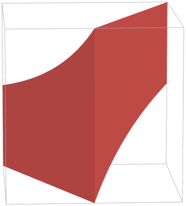
Totally nonnegative partial flag varieties of $\mathbb{C}^n$

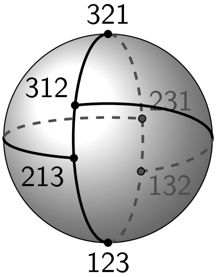
- For $1 \le k_1 < \cdots < k_r \le n-1$, let $\mathrm{Fl}_{k_1,\dots,k_r;n}(\mathbb{C})$ denote the set of all tuples

$$V_\bullet = (0 \subset V_{k_1} \subset \cdots \subset V_{k_r} \subset \mathbb{C}^n), \qquad \text{where } \dim(V_{k_i}) = k_i \text{ for all } i.$$

- e.g. $\mathrm{Fl}_{k;n}(\mathbb{C}) = \mathrm{Gr}_{k,n}(\mathbb{C})$, and $\mathrm{Fl}_{1,\dots,n-1;n}(\mathbb{C})$ is the *complete flag variety*.
- e.g. $\mathrm{Fl}^{\ge 0}_{1,2;3}$







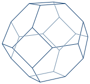
Theorem (Bloch–Karp (2023))

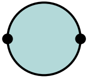
(i) If $V_\bullet \in \mathrm{Fl}^{\ge 0}_{k_1,\dots,k_r;n}$, then $\Delta_I(V_{k_i}) \ge 0$ for all i and all k_i-subsets I.

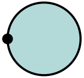
(ii) The converse holds for all $V_\bullet$ if and only if $k_1, \dots, k_r$ are consecutive.

Total positivity and regular CW complexes

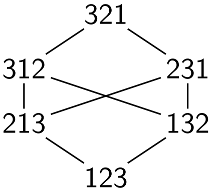
- A *regular CW complex* is a cell decomposition where every cell closure is homeomorphic to a closed ball (e.g. a polytope decomposed into its faces).


regular CW complex


regular CW complex


non-regular CW complex

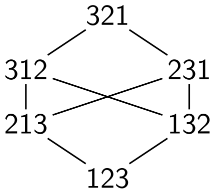
- Björner (1984): Every regular CW complex is uniquely determined by its closure poset (up to homeomorphism). Conversely, any poset which is *graded*, *thin*, and *shellable* is the poset of some regular CW complex.
- Edelman (1981): The Bruhat order on $\mathfrak{S}_n$ is graded, thin, and shellable.

$\mathfrak{S}_3 =$ 

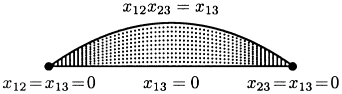
- Björner (1984): Is there a 'natural' regular CW complex with poset $\mathfrak{S}_n$?

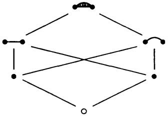
Regular CW complexes

- Fomin–Shapiro (2000) conjectured that $\operatorname{link}_{I_n}(U_n^{\geq 0})$ is a regular CW complex with poset $\mathfrak{S}_n$. This was proved by Hersh (2014) in all Lie types.



$$\operatorname{link}_{I_3}(U_3^{\geq 0}) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : \begin{array}{l} a + c = 1, \\ \text{all minors} \geq 0 \end{array} \right\}$$





- Other (potential) examples of regular CW complexes:
  - totally nonnegative part of a toric variety (Sottile (2003));
  - toric cubes (Basu–Gabrielov–Vorobjov (2013));
  - $(G/P)_{\geq 0}$ (Galashin–Karp–Lam (2022), Bao–He (2024));
  - nonnegative matroid Schubert varieties (He–Simpson–Xie (2023));
  - amplituhedra and Grassmann polytopes (?);
  - totally nonnegative cluster varieties (?);
  - spaces of Lorentzian polynomials (?).

Thank you!