The Wronski map and real Schubert calculus

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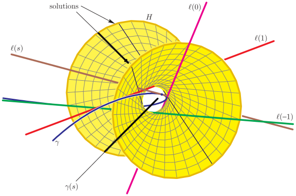

F. Sottile, “Frontiers of reality in Schubert calculus”

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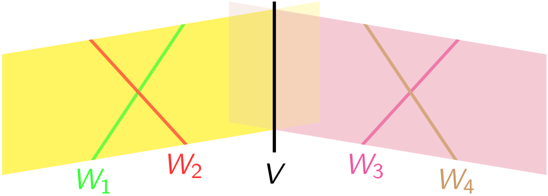
Schubert calculus (1886)

- Divisor Schubert problem: given $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{C})$,

$$\text{find all } V \in \mathrm{Gr}_{d,m}(\mathbb{C}) \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- e.g. $d = 2$, $m = 4$ (projectivized). Given 4 lines $W_i \subseteq \mathbb{CP}^3$, find all lines $V \subseteq \mathbb{CP}^3$ intersecting all 4. Generically, there are 2 solutions.



1	2
3	4

1	3
2	4

We can see the 2 solutions explicitly when two pairs of the lines intersect.

- If the W_i's are generic, the number of solutions V is the number of *standard Young tableaux* of rectangular shape $d \times (m - d)$.
- Fulton (1984): "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems."

Shapiro–Shapiro conjecture

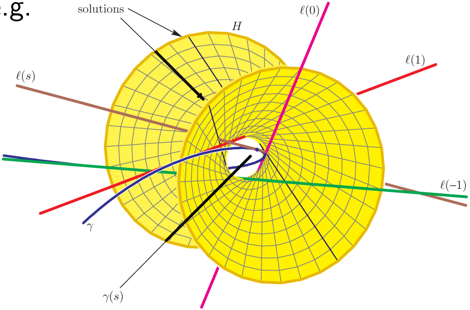
- Do there exist Schubert problems with all real solutions?

Shapiro–Shapiro conjecture (1993)

Let $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ *osculate the moment curve* $\gamma(t) := (\frac{t^{m-1}}{(m-1)!}, \frac{t^{m-2}}{(m-2)!}, \ldots, t, 1)$ *at real points. Then*

$$\textit{there exist } f^{\square} \textit{ real } V \in \mathrm{Gr}_{d,m}(\mathbb{R}) \textit{ such that } V \cap W_i \neq \{0\} \textit{ for all } i.$$

- e.g.



F. Sottile, "Frontiers of reality in Schubert calculus"

- This Schubert problem arises in the study of linear series in algebraic geometry, differential equations, and pole placement problems in control theory.
- Bürgisser, Lerario (2020): a uniformly random Schubert problem over $\mathbb{R}$ has $\approx\sqrt{f^{\square}}$ real solutions.

Shapiro–Shapiro conjecture and secant conjecture

- Sottile (1999) tested the conjecture and proved it asymptotically.
- Eremenko–Gabrielov (2002): cases $d \leq 2$, $m - d \leq 2$.
- Mukhin–Tarasov–Varchenko (2009): full conjecture via the *Bethe ansatz*.
- Levinson–Purbhoo (2021): topological proof of the full conjecture.
- Vakil (2006): reality of Grassmannian Schubert calculus.

Secant conjecture, divisor form (Sottile (2003))

Let $W_1, \dots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ be secant to the moment curve $\gamma(t)$ along non-overlapping real intervals. Then

$$\text{there exist } f^{\square} \text{ real } V \in \mathrm{Gr}_{d,m}(\mathbb{R}) \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- Eremenko–Gabrielov–Shapiro–Vainshtein (2006): case $m - d \leq 2$.

Theorem (Karp–Purbhoo (2023))

The divisor form of the secant conjecture is true.

The Wronskian

- The *Wronskian* of d functions $f_1, \dots, f_d : \mathbb{C} \to \mathbb{C}$ is

$$\mathrm{Wr}(f_1, \dots, f_d) := \det \begin{bmatrix} f_1 & \cdots & f_d \\ f_1' & \cdots & f_d' \\ \vdots & \ddots & \vdots \\ f_1^{(d-1)} & \cdots & f_d^{(d-1)} \end{bmatrix}.$$

- e.g. $\mathrm{Wr}(f, g) = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - f'g = f^2(\frac{g}{f})'$.

- $\mathrm{Wr}(f_1, \dots, f_d) \not\equiv 0$ if and only if $f_1, \dots, f_d$ are linearly independent. Then $\mathrm{Wr}(V)$ is well-defined up to a scalar, where $V := \langle f_1, \dots, f_d \rangle$.

- The zeros of $\mathrm{Wr}(V)$ are points in $\mathbb{C}$ where some nonzero $f \in V$ has a zero of order $\geq d$.

- The monic linear differential operator $\mathcal{L}$ of order d with kernel V is

$$\mathcal{L}(g) = \frac{\mathrm{Wr}(f_1, \dots, f_d, g)}{\mathrm{Wr}(f_1, \dots, f_d)} = g^{(d)} + \cdots .$$

Wronskian formulation of Shapiro–Shapiro conjecture

- We identify $\mathbb{C}^m$ with the space of polynomials of degree at most $m-1$:

$$\mathbb{C}^m \leftrightarrow \mathbb{C}_{m-1}[u], \quad (a_1, \ldots, a_m) \leftrightarrow a_1 + a_2 u + a_3 \tfrac{u^2}{2} + \cdots + a_m \tfrac{u^{m-1}}{(m-1)!}.$$

We obtain the *Wronski morphism* $\mathrm{Wr} : \mathrm{Gr}_{d,m}(\mathbb{C}) \to \mathbb{P}(\mathbb{C}_{d(m-d)}[u])$.

Lemma

Let $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ and $z \in \mathbb{C}$. Then $\mathrm{Wr}(V)$ is zero at $-z$ if and only if V nontrivially intersects the codimension-d osculating subspace to γ at z.

Shapiro–Shapiro conjecture (Mukhin–Tarasov–Varchenko (2009))

Let $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$. If all complex zeros of $\mathrm{Wr}(V)$ are real, then V is real.

- e.g. If $\mathrm{Wr}(V) := (u+z_1)(u+z_2)$, the two solutions $V \in \mathrm{Gr}_{2,4}(\mathbb{C})$ are

$$\left\langle 1, z_1 z_2 u + \tfrac{(z_1+z_2)u^2}{2} + \tfrac{u^3}{3} \right\rangle \quad \text{and} \quad \left\langle (u+z_1)^2, (u+z_2)^2 \right\rangle.$$

Disconjugacy and positivity conjectures

Disconjugacy conjecture (Eremenko (2015))

Let $V \in \mathrm{Gr}_{d,m}(\mathbb{R})$ be such that all complex zeros of $\mathrm{Wr}(V)$ are real, and let $K \subseteq \mathbb{R}$ be an interval avoiding the zeros of $\mathrm{Wr}(V)$. Then every nonzero real $f \in V$ has at most $d-1$ zeros in K.

- Eremenko (2015): this implies the divisor form of the secant conjecture.

Positivity conjecture (Mukhin–Tarasov (2017), Karp (2021))

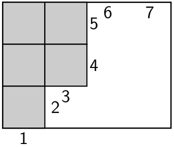
Let $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$. If all complex zeros of $\mathrm{Wr}(V)$ are ≤ 0, then $V \in \mathrm{Gr}_{d,m}^{\geq 0}$.

- Karp (2023): this is equivalent to the disconjugacy conjecture. Proof of the forward direction: by an action of $\mathrm{SL}_2(\mathbb{R})$, we can assume $K = (0, \infty)$. If the zeros of $\mathrm{Wr}(V)$ are real and avoid K, then the positivity conjecture implies $V \in \mathrm{Gr}_{d,m}^{\geq 0}$. By the Gantmakher–Krein theorem, every $f \in V$ has $\leq d-1$ sign changes, so it has $\leq d-1$ zeros in $(0, \infty)$ by Descartes's rule.
- Karp–Purbhoo (2023): The positivity conjecture is true.

Universal Plücker coordinates for the Wronski map

- We index Plücker coordinates by partitions inside a $d \times (m-d)$ rectangle.

$$\lambda = (2,2,1)$$
$$|\lambda| = 5$$
$$d = 3, m = 7$$



$$\leftrightarrow \quad I_\lambda = \{2,4,5\}$$

Theorem (Karp–Purbhoo (2023))

Let $z_1, \ldots, z_n \in \mathbb{C}$. There exist linear operators $\beta_\lambda(z_1, \ldots, z_n)$ such that:

(i) The β_λ's commute and satisfy the Plücker relations.

(ii) The common eigenspaces E of the β_λ's correspond to $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ with $\mathrm{Wr}(V) = (u+z_1)\cdots(u+z_n)$, via $\beta_\lambda|_E \mapsto \Delta_{I_\lambda}(V)$.

(iii) If $z_1, \ldots, z_n \geq 0$, then the β_λ's are positive semidefinite.

$$\beta_\lambda := \sum_{\substack{X \subseteq \{1,\ldots,n\}, \\ |X| = |\lambda|}} \Big(\prod_{i \notin X} z_i\Big) \sum_{\pi \in \mathfrak{S}_X} \chi^\lambda(\pi)\pi \in \mathbb{C}[\mathfrak{S}_n]$$

Example: $d = 2$, $m = 4$, $n = 2$

$$\beta_\lambda := \sum_{\substack{X \subseteq \{1,\dots,n\},\\ |X| = |\lambda|}} \Big(\prod_{i \notin X} z_i\Big) \sum_{\pi \in \mathfrak{S}_X} \chi^\lambda(\pi)\pi \in \mathbb{C}[\mathfrak{S}_n]$$

- Write $\mathfrak{S}_2 = \{e, \sigma\}$, where e is the identity and $\sigma = (1\ \ 2)$. We have

$$\beta_\varnothing = z_1 z_2 e, \qquad \beta_{\square} = (z_1 + z_2)e, \qquad \beta_{\square\square} = e + \sigma, \qquad \beta_{\substack{\square\\\square}} = e - \sigma,$$

and $\beta_\lambda = 0$ if $|\lambda| > 2$. The Plücker relation is $\beta_{\square}\beta_{\substack{\square\square\\\square}} = \beta_\varnothing \beta_{\substack{\square\square\\\square\square}} + \beta_{\substack{\square\\\square}}\beta_{\square\square}$.

- On the sign eigenspace $E = M^{\substack{\square\\\square}}$, the eigenvalues are

$$\beta_\varnothing \rightsquigarrow z_1 z_2, \qquad \beta_{\square} \rightsquigarrow z_1 + z_2, \qquad \beta_{\square\square} \rightsquigarrow 0, \qquad \beta_{\substack{\square\\\square}} \rightsquigarrow 2.$$

These are the Plücker coordinates of

$$V = \begin{bmatrix} \frac{z_1+z_2}{2} & 1 & 0 & 0 \\ -z_1 z_2 & 0 & 2 & 0 \end{bmatrix} = \Big\langle \tfrac{z_1+z_2}{2} + u,\ -z_1 z_2 + u^2 \Big\rangle \in X^{\substack{\square\\\square}} \subseteq \mathrm{Gr}_{2,4}(\mathbb{C}).$$

We can check that $\mathrm{Wr}(V) = (u + z_1)(u + z_2)$.

Proof 1: KP hierarchy

- The key to the proof is showing that the β_λ's satisfy the Plücker relations.
- The KP equation models shallow waves. It is the first equation in the *KP hierarchy*, whose solutions are symmetric functions $\tau(\mathbf{x})$ in $\mathbf{x} = (x_1, x_2, \dots)$ satisfying *Hirota's identity*

$$[t^{-1}]\big(B_{\mathbf{x}}(t)\tau(\mathbf{x}) \cdot B_{\mathbf{y}}^{\perp}(t^{-1})\tau(\mathbf{y})\big) = 0.$$

Here $\cdot^{\perp}$ denotes the adjoint with respect to $\langle \cdot, \cdot \rangle$ (so $p_k(\mathbf{x})^{\perp} = k\frac{\partial}{\partial p_k(\mathbf{x})}$), and

$$B_{\mathbf{x}}(t) := H_{\mathbf{x}}(t)E_{\mathbf{x}}^{\perp}(-t^{-1}), \quad H_{\mathbf{x}}(t) := \sum_{k \geq 0} h_k(\mathbf{x})t^k, \quad E_{\mathbf{x}}(t) := \sum_{k \geq 0} e_k(\mathbf{x})t^k.$$

- Sato (1981): $\tau(\mathbf{x})$ satisfies Hirota's identity if and only if its coefficients in the Schur basis $s_\lambda(\mathbf{x})$ satisfy the Plücker relations.
- Karp–Purbhoo (2023): $\sum_\lambda \beta_\lambda s_\lambda(\mathbf{x})$ satisfies Hirota's identity.

Proof 2: higher Gaudin Hamiltonians

- Define

$$T_\lambda := (z_1 + \mathbf{d}_1) \cdots (z_n + \mathbf{d}_n) s_\lambda(h) \in \mathrm{End}\big((\mathbb{C}^d)^{\otimes n}\big),$$

where:

- h is a $d \times d$ matrix;
- $s_\lambda(h)$ is the Schur polynomial evaluated at the eigenvalues of h; and
- $\mathbf{d}_i$ is the derivative with respect to h^{T} acting in the ith tensor factor.

Theorem (Alexandrov–Leurent–Tsuboi–Zabrodin (2014))

The T_λ's pairwise commute and satisfy the Plücker relations.

Theorem (Karp–Mukhin–Tarasov (2024))

(i) We have $T_\lambda|_{h=0} = \beta_\lambda$.

(ii) If $z_1, \ldots, z_n \geq 0$ and h is positive semidefinite, then so is T_λ.

- Part (ii) gives a positivity theorem for spaces $\langle e^{h_1 u} f_1(u), \ldots, e^{h_d u} f_d(u) \rangle$, where $h_1, \ldots, h_d$ are the eigenvalues of h and the f_i's are polynomials.

Computing with higher Gaudin Hamiltonians

- e.g. $d=2$, $n=2$. Let us verify that $T_{\square\square}|_{h=0} = \beta_{\square\square}$, i.e.,

$$\mathbf{d}_2\,\mathbf{d}_1\,s_{\square\square}(h) = e + \sigma \in \mathrm{End}\big((\mathbb{C}^2)^{\otimes 2}\big), \qquad \text{where } \mathfrak{S}_2 = \{e, \sigma\}.$$

- Denote $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so that $\mathbf{d}\,\phi(h) = \begin{bmatrix} \partial_a\phi & \partial_c\phi \\ \partial_b\phi & \partial_d\phi \end{bmatrix}$. We have

$$s_{\square\square}(h) = \frac{p_{\substack{\square\\\square}}(h) + p_{\square\square}(h)}{2} = \frac{\mathrm{Tr}(h)^2 + \mathrm{Tr}(h^2)}{2} = a^2 + d^2 + ad + bc.$$

- Then

$$\mathbf{d}_1\,s_{\square\square}(h) = \begin{bmatrix} 2a+d & b \\ c & a+2d \end{bmatrix},$$

$$\mathbf{d}_2\,\mathbf{d}_1\,s_{\square\square}(h) = \mathbf{d}_2\left((2a+d)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (a+2d)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= (v \otimes w \mapsto v \otimes w + w \otimes v) = e + \sigma.$$

Future directions

- Further explore the connection to the KP hierarchy.
- Find necessary and sufficient inequalities on the Plücker coordinates of V for all complex zeros of $\mathrm{Wr}(V)$ to be nonpositive, generalizing the Aissen–Schoenberg–Whitney theorem in the case $\dim(V) = 1$. (The positivity conjecture implies the inequalities $\Delta_\lambda(V) \geq 0$ are necessary.)
- What happens to the higher Gaudin Hamiltonian T_λ if s_λ is replaced by a different symmetric function?
- Address generalizations and variations of the Shapiro–Shapiro conjecture: the discriminant conjecture, the general form of the secant conjecture, the monotone conjecture, the total reality conjecture for convex curves, ...

Thank you!