q-Whittaker functions, finite fields, and Jordan forms

Slides available at `snkarp.github.io`

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Schur functions

- A *partition* λ is a weakly-decreasing sequence of nonnegative integers.

- e.g. $\lambda = (4,4,1) = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline \\ \cline{1-1} \end{array}$ $\qquad T = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 & 4 \\ \hline 4 & 4 & 8 & 8 \\ \hline 5 \\ \cline{1-1} \end{array}$

- A *semistandard tableau* T is a filling of λ with positive integers which is weakly increasing across rows and strictly increasing down columns.

Definition (Schur function)

$$s_\lambda(x_1, x_2, \dots) := \sum_T \mathbf{x}^T,$$

where the sum is over all semistandard tableaux T of shape λ.

- $s_\lambda(\mathbf{x})$ is symmetric in the variables x_i.

Schur functions

- e.g. $s_{(2,1)}(x_1, x_2, x_3) =$

$$
\begin{array}{cccccccccccccccc}
x_1^2x_2 & + & x_1^2x_3 & + & x_1x_2^2 & + & x_1x_2x_3 & + & x_1x_2x_3 & + & x_1x_3^2 & + & x_2^2x_3 & + & x_2x_3^2 \\
\begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix} & & \begin{smallmatrix} 1 & 1 \\ 3 & \end{smallmatrix} & & \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix} & & \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} & & \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} & & \begin{smallmatrix} 1 & 3 \\ 3 & \end{smallmatrix} & & \begin{smallmatrix} 2 & 2 \\ 3 & \end{smallmatrix} & & \begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix}
\end{array}
$$

- Schur functions appear in many contexts; for example, they:
  - form an *orthonormal basis* for the algebra of symmetric functions in $\mathbf{x}$;
  - are characters of the *irreducible polynomial representations* of $\mathrm{GL}_n(\mathbb{C})$;
  - give the values of the *irreducible characters* of the symmetric group S_n, when expanded in terms of power sum symmetric functions;
  - are representatives for *Schubert classes* in the cohomology ring of the Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{C})$;
  - define the *Schur processes* of Okounkov and Reshetikhin (2003).

Cauchy identity

Theorem (Cauchy)

$$\prod_{i,j\geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y})$$

- The identity is equivalent to the orthonormality of the Schur functions. It also gives the partition function for the Schur processes.
- The left-hand side counts *nonnegative-integer matrices*, and the right-hand side counts *pairs of semistandard tableaux of the same shape*.
- e.g. Taking the coefficient of $x_1 x_2 y_1 y_2$ on each side gives

$$\begin{array}{ccccccc} 1 & + & 1 & = & 1 & + & 1 \\ 12 & & 21 & & \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right) & & \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right)\end{array}$$

Burge correspondence (1974)

- The *Burge correspondence* (also known as *column Robinson–Schensted–Knuth*) is a bijection

$$M \mapsto (\mathrm{P}(M), \mathrm{Q}(M))$$

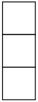
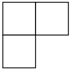
between nonnegative-integer matrices and pairs of semistandard tableaux of the same shape. It proves the Cauchy identity for Schur functions.

- $\mathrm{P}(M)$ is obtained via *column insertion* and $\mathrm{Q}(M)$ via *recording*.
- e.g. $w = 25143$

$$\begin{array}{|c|}\hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 5 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline \end{array} \qquad \underset{\mathrm{P}(w)}{\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}} \qquad \underset{\mathrm{Q}(w)}{\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}}$$

Nilpotent matrices

- An $n \times n$ matrix N over $\mathbb{k}$ is *nilpotent* if some power of N is zero. Such an N can be conjugated over $\mathbb{k}$ into *Jordan form*. Let $\mathrm{JF}^{\top}(N)$ be the *transpose* of the partition given by the sizes of the Jordan blocks.

- e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto$  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix} \mapsto$  $\begin{bmatrix} \boxed{0} & 0 & 0 \\ 0 & \boxed{0} & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix} \mapsto$ 

- Algebraically, $\mathrm{JF}^{\top}(N)$ is the partition λ given by

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i = \dim(\ker(N^i)) \quad \text{for all } i.$$

Theorem (Gansner (1981))

Let N be a generic $n \times n$ strictly upper-triangular matrix, where $N_{i,j} = 0$ for all inversions (i,j) of w^{-1}. Then $\mathrm{P}(w)$ and $\mathrm{Q}(w)$ can be read off from the Jordan forms of the leading submatrices of N and $w^{-1}Nw$.

Burge correspondence via Jordan forms

- e.g. $w = 25143$ $\qquad N = \begin{bmatrix} 0 & 0 & a & b & 0 \\ 0 & 0 & c & d & e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\qquad (a, b, c, d, e \in \Bbbk \text{ generic})$

P(w):

$$\begin{array}{cc} & \begin{matrix}1\end{matrix} \\ \begin{matrix}1\end{matrix} & \begin{bmatrix}0\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}1 & 2\end{matrix} \\ \begin{matrix}1\\2\end{matrix} & \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}1 & 2 & 3\end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{bmatrix}0 & 0 & a\\0 & 0 & c\\0 & 0 & 0\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}0 & 0 & a & b\\0 & 0 & c & d\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4 & 5\end{matrix} \\ \begin{matrix}1\\2\\3\\4\\5\end{matrix} & \begin{bmatrix}0 & 0 & a & b & 0\\0 & 0 & c & d & e\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}\end{array}$$

$$\begin{array}{|c|}\hline \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

Q(w):

$$\begin{array}{cc} & \begin{matrix}2\end{matrix} \\ \begin{matrix}2\end{matrix} & \begin{bmatrix}0\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}2 & 5\end{matrix} \\ \begin{matrix}2\\5\end{matrix} & \begin{bmatrix}0 & e\\0 & 0\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}2 & 5 & 1\end{matrix} \\ \begin{matrix}2\\5\\1\end{matrix} & \begin{bmatrix}0 & e & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}2 & 5 & 1 & 4\end{matrix} \\ \begin{matrix}2\\5\\1\\4\end{matrix} & \begin{bmatrix}0 & e & 0 & d\\0 & 0 & 0 & 0\\0 & 0 & 0 & b\\0 & 0 & 0 & 0\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}2 & 5 & 1 & 4 & 3\end{matrix} \\ \begin{matrix}2\\5\\1\\4\\3\end{matrix} & \begin{bmatrix}0 & e & 0 & d & c\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & b & a\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}\end{array}$$

$$\begin{array}{|c|}\hline \\ \hline\end{array} \qquad \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

Flag variety

- A *complete flag* F in $\Bbbk^n$ is a sequence of nested subspaces

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = \Bbbk^n, \qquad \dim(F_i) = i \text{ for all } i.$$

- An $n \times n$ (nilpotent) matrix N is *strictly compatible* with F if

$$N(F_i) \subseteq F_{i-1} \quad \text{for all } i.$$

- The matrix N in Gansner's theorem is precisely one which is strictly compatible with two complete flags F and F' defined by

$$F_i := \langle \mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_i \rangle \quad \text{and} \quad F'_j := \langle \mathrm{e}_{w(1)}, \mathrm{e}_{w(2)}, \ldots, \mathrm{e}_{w(j)} \rangle.$$

The two sequences of matrices in the theorem are $(N|_{F_i})_{i=1}^n$ and $(N|_{F'_j})_{j=1}^n$.

- More generally, we can take any pair of flags (F, F') with *relative position* w, denoted $F \xrightarrow{w} F'$. The relative position records $\dim(F_i \cap F'_j)$ for all i and j, or alternatively, the *Schubert cell* of F' relative to F.

Burge correspondence via flags

Theorem (Steinberg (1976, 1988), Spaltenstein (1982), Rosso (2012))

Fix partial flags F and F' with $F \xrightarrow{M} F'$. Let N be a generic nilpotent matrix strictly compatible with both F and F'. Then

$$\mathrm{P}(M) = \mathrm{JF}^{\top}(N; F) \quad \textit{and} \quad \mathrm{Q}(M) = \mathrm{JF}^{\top}(N; F').$$

- If $F \xrightarrow{w} F'$, then $F' \xrightarrow{w^{-1}} F$. This implies the symmetry

$$\mathrm{P}(w^{-1}) = \mathrm{Q}(w).$$

- What happens when $\Bbbk$ is a *finite* field, and we consider *all* choices of N (not necessarily generic)?

q-Whittaker functions

- Define $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$.

Definition (q-Whittaker function)

$$W_\lambda(x_1, x_2, \dots; q) := \sum_T \mathrm{wt}_q(T)\mathbf{x}^T,$$

where the sum is over all semistandard tableaux T of shape λ.

- $W_\lambda(\mathbf{x}; q)$ is symmetric in the variables x_i, and specializes to $s_\lambda(\mathbf{x})$ when $q = 0$. We obtain the $\mathfrak{gl}_n$-*Whittaker functions* as a certain $q \to 1$ limit.

- e.g. $T = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 7 \\ \hline 6 \\ \cline{1-1} \end{array}$ $\qquad \mathrm{wt}_q(T) = [1]_q[2]_q[1]_q[2]_q[2]_q[1]_q[2]_q = (1+q)^4$

- We have the following specializations:

$$W_\lambda(\mathbf{x}; q) = P_\lambda(\mathbf{x}; q, 0) = q^{\deg(\widetilde{H}_\lambda)}\omega(\widetilde{H}_\lambda(\mathbf{x}; 1/q, 0)), \quad W_\lambda(\mathbf{x}; 1) = e_{\lambda^\top}(\mathbf{x}).$$

q-Cauchy identity

Theorem (Macdonald (1995))

$$\prod_{i,j\geq 1}\prod_{d\geq 0}\frac{1}{1-x_iy_jq^d}=\sum_{\lambda}\frac{(1-q)^{-\lambda_1}}{\prod_{i\geq 1}[\lambda_i-\lambda_{i+1}]_q!}W_\lambda(\mathbf{x};q)W_\lambda(\mathbf{y};q)$$

- This gives the partition function for the *q-Whittaker processes*, a special case of the *Macdonald processes* of Borodin and Corwin (2014).
- e.g. Taking the coefficient of $x_1x_2y_1y_2$ on each side gives

$$(1-q)^{-2} \;+\; (1-q)^{-2} \;=\; (1-q)^{-1} \;+\; (1-q)^{-2}(1+q)$$

$$12 \qquad\qquad 21 \qquad\qquad \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\,,\,\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right) \qquad\qquad \left(\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\,,\,\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\right)$$

$1-q$ $\qquad$ 1

q

q-Burge correspondence

- e.g. $w = 12 \qquad N = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \qquad (a \in \mathbb{F}_{1/q})$

$$\mathrm{P}(w): \quad \begin{array}{cc} & 1 \\ 1 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \quad \begin{array}{ccc} & 1 & 2 \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \end{array} \qquad \mathrm{Q}(w): \quad \begin{array}{cc} & 1 \\ 1 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \quad \begin{array}{ccc} & 1 & 2 \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \end{array}$$

$$a \neq 0: \quad \square \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \square \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \mathbb{P} = 1 - q$$

$$a = 0: \quad \square \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \square \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \mathbb{P} = q$$

- e.g. $w = \overset{\frown}{21} \qquad N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\mathrm{P}(w): \quad \begin{array}{cc} & 1 \\ 1 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \quad \begin{array}{ccc} & 1 & 2 \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \qquad \mathrm{Q}(w): \quad \begin{array}{cc} & 2 \\ 2 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \quad \begin{array}{ccc} & 2 & 1 \\ \begin{array}{c} 2 \\ 1 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\square \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \square \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \mathbb{P} = 1$$

q-Burge correspondence

- Fix partial flags $F \xrightarrow{M} F'$ over $\mathbb{F}_{1/q}$. For semistandard tableaux T and T' of the same shape, let N denote a uniformly random nilpotent matrix strictly compatible with both F and F'. Define

$$\mathrm{p}_M(T, T') := \mathbb{P}(\mathrm{JF}^\top(N; F) = T \text{ and } \mathrm{JF}^\top(N; F') = T'), \qquad (*)$$

Theorem (Karp, Thomas (2022))

(i) The maps $\mathrm{p}_M(\cdot, \cdot)$ define a probabilistic bijection proving the Cauchy identity for q-Whittaker functions, called the q-Burge correspondence.
(ii) The bijection converges to the classical Burge correspondence as $q \to 0$.

- The inverse probabilities are given by $(*)$ for N fixed and (F, F') random.

Problem

Is $\mathrm{p}_M(T, T')$ a rational function of q? (If so, it is a polynomial.)

- Two other probabilistic bijections were given by Matveev and Petrov (2017), using q-analogues of row and column insertion.

Quiver representations and the preprojective algebra

- Consider a path quiver with a unique sink:

$$Q = \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & {\color{red}\bullet} & \bullet \arrow[l] & \bullet \arrow[l] \end{tikzcd}$$

- A *representation* V of Q is an assignment of a vector space to each vertex and a linear map to each arrow, e.g.,

$$V = \begin{tikzcd} \Bbbk \arrow[r, "{\begin{bmatrix} -1 \end{bmatrix}}"] & \Bbbk \arrow[r, "{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}"] & {\color{red}\Bbbk^2} & \Bbbk^2 \arrow[l, "{\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}}"'] & 0 \arrow[l, "0"'] \end{tikzcd}$$

- We will only consider V where every linear map is injective. Isomorphism classes of such V are indexed by nonnegative-integer matrices M.

- We now decorate V with a linear map for the reverse of each arrow, such that a relation holds for every vertex:

$$\begin{tikzcd} {} \arrow[r, bend left, "\alpha"] & \bullet \arrow[l, bend left, "\gamma"] \arrow[r, bend left, "\beta"] & {} \arrow[l, bend left, "\delta"] \end{tikzcd} \qquad \alpha \circ \gamma \pm \delta \circ \beta = 0$$

This defines a module $V^\sharp$ over the *preprojective algebra* of Q.

Socle filtration

- Up to isomorphism, $V^\sharp$ is given (non-uniquely) by a triple (F, F', N):

$$V^\sharp = \begin{tikzcd} F_1 \arrow[r, bend left, "\mathrm{id}"] & F_2 \arrow[l, bend left, "N"] \arrow[r, bend left, "\mathrm{id}"] & F_3 = F_3' \arrow[l, bend left, "N"] \arrow[r, bend right, "-N"'] & F_2' \arrow[l, bend right, "\mathrm{id}"'] \arrow[r, bend right, "-N"'] & F_1' \arrow[l, bend right, "\mathrm{id}"'] \end{tikzcd}$$

- The *socle filtration* of $V^\sharp$ corresponds precisely to the pair of tableaux

$$(T, T') = (\mathrm{JF}^\top(N; F), \mathrm{JF}^\top(N; F')).$$

- e.g.

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 2 & \end{array} \quad \longleftrightarrow \quad \left(\begin{array}{ccc} & 1 & \\ 2 & & 1 \end{array} , \begin{array}{ccc} & 1 & \\ 2 & & 1 \end{array} \right)$$

$$V^\sharp = \begin{tikzcd} \langle e_1 \rangle \arrow[r, bend left, "\mathrm{id}"] & \mathbb{k}^3 \arrow[l, bend left, "{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}"] \arrow[r, bend right, "{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}"'] & \langle e_1 \rangle \arrow[l, bend right, "\mathrm{id}"'] \end{tikzcd} \quad \longleftrightarrow \quad \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array} \right)$$

Counting isomorphism classes

- The q-Burge correspondence implies enumerative results about such modules $V^\sharp$. For example:

Theorem (Karp, Thomas (2022))

Let (T, T') be a pair of semistandard tableaux of shape λ, and let $\mathbf{d}$ be a dimension vector of Q. Then

$$\sum_{[V^\sharp]} \frac{1}{|\mathrm{Aut}(V^\sharp)|} = \frac{q^{c(\mathbf{d})}(1-q)^{-\lambda_1}}{\prod_{i \geq 1} [\lambda_i - \lambda_{i+1}]_q!} \mathrm{wt}_q(T) \, \mathrm{wt}_q(T'),$$

where the sum is over all isomorphism classes $[V^\sharp]$ of modules $V^\sharp$ over $\mathbb{F}_{1/q}$ with dimension vector $\mathbf{d}$ and socle filtration corresponding to (T, T').

Thank you!