Sign variation, the Grassmannian, and total positivity

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Alternating curves

Proposition

Let $f : [0,1] \to \mathbb{R}^k$ be a continuous curve. Then no hyperplane through 0 contains k points on the curve iff the determinants

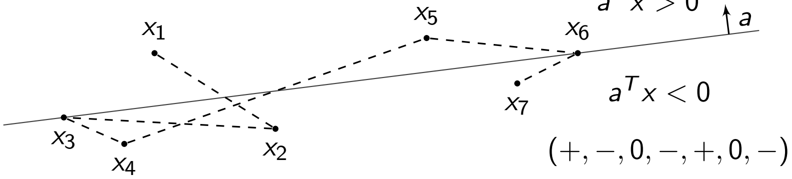
$$\det[\, f(t_1) \mid \cdots \mid f(t_k)\,] \qquad (0 \le t_1 < \cdots < t_k \le 1)$$

are either all positive or all negative.

Proof

Since $\{(t_1, \cdots, t_k) \in \mathbb{R}^k : 0 \le t_1 < \cdots < t_k \le 1\} \subseteq \mathbb{R}^k$ is connected, its image $\{\det[\, f(t_1) \mid \cdots \mid f(t_k)\,] : 0 \le t_1 < \cdots < t_k \le 1\} \subseteq \mathbb{R}$ is connected.

- How can we discretize this result?

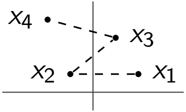


Alternating curves

Theorem (Gantmakher, Krein (1950); Schoenberg, Whitney (1951))

Let $x_1, \cdots, x_n \in \mathbb{R}^k$ span $\mathbb{R}^k$. Then the following are equivalent:
(i) the piecewise-linear path $x_1, \cdots, x_n$ crosses any hyperplane through 0 at most $k-1$ times;
(ii) the sequence $(a^T x_1, \cdots, a^T x_n)$ changes sign at most $k-1$ times for all $a \in \mathbb{R}^n$; and
(iii) the $k \times k$ minors of the $k \times n$ matrix $[x_1|\cdots|x_n]$ are either all nonnegative or all nonpositive.

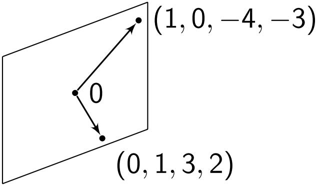
- e.g.



- The set of such point configurations $(x_1, \cdots, x_n)$, modulo linear automorphisms of $\mathbb{R}^k$, is the *totally nonnegative Grassmannian*.
- Can we characterize the maximum number of hyperplane crossings of the path $x_1, \cdots, x_n$ in terms of the $k \times k$ minors of $[x_1|\cdots|x_n]$?

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces V of $\mathbb{R}^n$.

$V :=$ 

$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}$$

$$\Delta_{\{1,2\}} = 1, \Delta_{\{1,3\}} = 3, \Delta_{\{1,4\}} = 2, \Delta_{\{2,3\}} = 4, \Delta_{\{2,4\}} = 3, \Delta_{\{3,4\}} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for $I \in \binom{[n]}{k}$ let $\Delta_I(V)$ be the $k \times k$ minor of V with columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to multiplication by a global nonzero constant.
- We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if $\Delta_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$, and *totally positive* if $\Delta_I(V) > 0$ for all $I \in \binom{[n]}{k}$. Denote the set totally nonnegative V by $\mathrm{Gr}_{k,n}^{\geq 0}$, and the set of totally positive V by $\mathrm{Gr}_{k,n}^{>0}$.

Sign variation

- For $v \in \mathbb{R}^n$, let $\text{var}(v)$ be the number of sign changes in the sequence $(v_1, v_2, \cdots, v_n)$, ignoring any zeros.

$$\text{var}(1, -4, 0, -3, 6, 0, -1) = \text{var}(1, -4, -3, 6, -1) = 3$$

Similarly, let $\overline{\text{var}}(v)$ be the maximum of $\text{var}(w)$ over all $w \in \mathbb{R}^n$ obtained from v by changing zero components of w.

$$\overline{\text{var}}(1, -4, 0, -3, 6, 0, -1) = 5$$

Theorem (Gantmakher, Krein (1950))

Let $V \in \text{Gr}_{k,n}$.
(i) V *is totally nonnegative iff* $\text{var}(v) \leq k - 1$ *for all* $v \in V$.
(ii) V *is totally positive iff* $\overline{\text{var}}(v) \leq k - 1$ *for all nonzero* $v \in V$.

- e.g. $\begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \text{Gr}_{2,4}^{>0}$.
- Note that every $V \in \text{Gr}_{k,n}$ contains a vector v with $\text{var}(v) \geq k - 1$.

A history of sign variation and total positivity

- Descartes's rule of signs (1637): The number of positive real zeros of a real polynomial $\sum_{i=0}^{n} a_i t^i$ is at most $\mathrm{var}(a_0, a_1, \cdots, a_n)$.
- Pólya (1912) asked when a linear map $A : \mathbb{R}^k \to \mathbb{R}^n$ *diminishes variation*, i.e. satisfies $\mathrm{var}(Ax) \le \mathrm{var}(x)$ for all $x \in \mathbb{R}^k$. Schoenberg (1930) showed that an injective A diminishes variation iff for $j = 1, \cdots, k$, all nonzero $j \times j$ minors of A have the same sign.

formations. **The problem of characterizing such transformations was attacked by Schoenberg in 1930 with only partial success**

- Gantmakher, Krein (1935): The eigenvalues of a *totally positive* square matrix (whose minors are all positive) are real, positive, and distinct.
- Gantmakher, Krein (1950): *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems* (Russian), 2nd ed., 359pp.

22825 (AEC-tr-4481) OSCILLATION MATRICES AND KERNELS AND SMALL VIBRATIONS OF MECHANICAL SYSTEMS. Second Edition Corrected and Expanded. F. R. Gantmakher and M. G. Krein. Translated from a Publication of the State Publishing House for Technical-Theoretical Literature, Moscow-Leningrad, 1950. 414p.

A natural mathematical base is proposed for the investigation of the so-called oscillation properties of small harmonic oscillations of linear elastic continua, such as, transverse oscillations of strings, rods, and multiple-span beams, and torsional oscillations of shafts. The book is

A history of sign variation and total positivity

- Whitney (1952): The totally positive matrices are dense in the totally nonnegative matrices.
- Aissen, Schoenberg, Whitney (1952): Let $r_1, \cdots, r_n \in \mathbb{C}$. Then $r_1, \cdots, r_n$ are all nonnegative reals iff $s_\lambda(r_1, \cdots, r_n) \geq 0$ for all partitions λ.
- Karlin (1968): *Total Positivity, Volume I*, 576pp.
- Lusztig (1994) constructed a theory of total positivity for G and G/P.

> One of the main tools in our study of $G_{\geq 0}$ and $G_{>0}$ is the theory of canonical bases in [L1]. Thus, our proof of the fact that $G_{\geq 0}$ is closed in G (Theorem 4.3) is based on the positivity properties of the canonical bases (in the simply-laced case), proved in [L1],[L2], which is a non-elementary statement, depending ultimately on the Weil conjectures. The

Rietsch (1997) and Marsh, Rietsch (2004) developed the theory for G/P.

- Fomin and Zelevinsky (2000s) introduced cluster algebras.
- Postnikov (2006) and others studied the combinatorics of $\mathrm{Gr}_{k,n}^{\geq 0}$.
- Kodama, Williams (2014): A τ-function $\tau = \sum_{I \in \binom{[n]}{k}} \Delta_I(V) s_{\lambda(I)}$ associated to $V \in \mathrm{Gr}_{k,n}$ gives a *regular* soliton solution to the KP equation iff V is totally nonnegative.

How close is a subspace to being totally positive?

- Can we determine $\max_{v\in V} \mathrm{var}(v)$ and $\max_{v\in V\setminus\{0\}} \overline{\mathrm{var}}(v)$ from the Plücker coordinates of V?

Theorem (Karp (2015))

Let $V \in \mathrm{Gr}_{k,n}$ and $s \geq 0$. Then $\overline{\mathrm{var}}(v) \leq k-1+s$ for all nonzero $v \in V$ iff

$$\overline{\mathrm{var}}((\Delta_{J\cup\{i\}}(V))_{i\notin J}) \leq s$$

for all $J \in \binom{[n]}{k-1}$ such that the sequence above is not identically zero.

- e.g. Let $V := \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}_{2,4}$ and $s := 1$. The fact that $\overline{\mathrm{var}}(v) \leq 2$ for all $v \in V \setminus \{0\}$ is equivalent to the fact that the sequences

$$(\Delta_{\{1,2\}}, \Delta_{\{1,3\}}, \Delta_{\{1,4\}}) = (2,1,1), \quad (\Delta_{\{1,3\}}, \Delta_{\{2,3\}}, \Delta_{\{3,4\}}) = (1,4,-6),$$

$$(\Delta_{\{1,2\}}, \Delta_{\{2,3\}}, \Delta_{\{2,4\}}) = (2,4,-8), \quad (\Delta_{\{1,4\}}, \Delta_{\{2,4\}}, \Delta_{\{3,4\}}) = (1,-8,-6)$$

each change sign at most once.

How close is a subspace to being totally nonnegative?

Theorem (Karp (2015))

Let $V \in \mathrm{Gr}_{k,n}$ and $s \geq 0$.
(i) If $\mathrm{var}(v) \leq k-1+s$ *for all* $v \in V$*, then*

$$\mathrm{var}((\Delta_{J \cup \{i\}}(V))_{i \notin J}) \leq s \qquad \text{for all } J \in \binom{[n]}{k-1}.$$

The converse holds if V is generic *(i.e.* $\Delta_I(V) \neq 0$ *for all I).*
(ii) We can perturb V into a generic W with $\max_{v \in V} \mathrm{var}(v) = \max_{v \in W} \mathrm{var}(v)$.

- e.g. Consider $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \overset{0.1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0.1 & 1 \end{bmatrix} \overset{0.01}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & 0.01 \\ 0 & 1 & 0.1 & 1.001 \end{bmatrix}$.

The 4 sequences of Plücker coordinates are

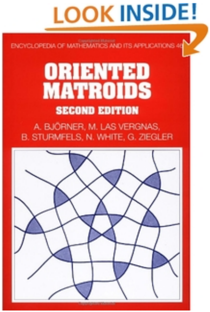
$(\Delta_{\{1,2\}}, \Delta_{\{1,3\}}, \Delta_{\{1,4\}}) = (1, \overset{0.1}{\cancel{0}}, \overset{1.001}{\cancel{1}})$, $\quad (\Delta_{\{1,3\}}, \Delta_{\{2,3\}}, \Delta_{\{3,4\}}) = (\overset{0.1}{\cancel{0}}, -1, 1)$,

$(\Delta_{\{1,2\}}, \Delta_{\{2,3\}}, \Delta_{\{2,4\}}) = (1, -1, \overset{-0.01}{\cancel{0}})$, $\quad (\Delta_{\{1,4\}}, \Delta_{\{2,4\}}, \Delta_{\{3,4\}}) = (\overset{1.001}{\cancel{1}}, \overset{-0.01}{\cancel{0}}, 1)$.

- Note: var is *increasing* while $\overline{\mathrm{var}}$ is *decreasing* with respect to genericity.

Oriented matroids

- An *oriented matroid* is a combinatorial abstraction of a real subspace, which records the Plücker coordinates up to sign, or equivalently the vectors up to sign.

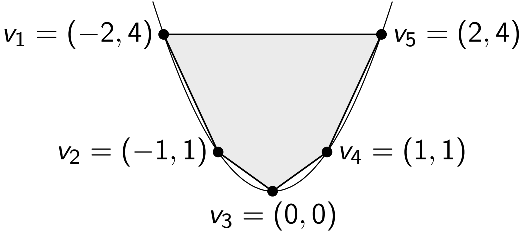


- These results generalize to oriented matroids.

Amplituhedra

- Let $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ be a linear map, and $Z_{\mathrm{Gr}} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ the map it induces on $\mathrm{Gr}_{k,n}^{\geq 0}$. In the case that all $(k+m) \times (k+m)$ minors of Z are positive, the image $Z_{\mathrm{Gr}}(\mathrm{Gr}_{k,n}^{\geq 0})$ is called a *(tree) amplituhedron*.
- e.g. Let $Z := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 1 & 1 & 4 \end{bmatrix}$ and $k := 1$. Then $Z_{\mathrm{Gr}}(\mathrm{Gr}_{1,5}^{\geq 0})$ equals

$$\left\{ \begin{matrix} (1 : -2a - b + d + 2e : \\ \quad 4a + b + c + d + 4e) \end{matrix} : \begin{matrix} a, b, c, d, e \geq 0, \\ a + b + c + d + e = 1 \end{matrix} \right\} \subseteq \mathbb{P}^2.$$



Amplituhedra

- When $k = 1$, amplituhedra are precisely *cyclic polytopes*. Cyclic polytopes achieve the maximum number of faces (in every dimension) in Stanley's upper bound theorem (1975).
- Lam (2015) proposed relaxing the positivity condition on Z, and called the more general class of images $Z_{\mathrm{Gr}}(\mathrm{Gr}_{k,n}^{\geq 0})$ *Grassmann polytopes*. When $k = 1$, Grassmann polytopes are precisely polytopes.
- Arkani-Hamed and Trnka (2013) introduced amplituhedra in order to study *scattering amplitudes*, which they compute as an integral over the amplituhedron $Z_{\mathrm{Gr}}(\mathrm{Gr}_{k,n}^{\geq 0})$ when $m = 4$.
- A *scattering amplitude* is a complex number whose modulus squared is the probability of observing a certain *scattering process*, e.g. a process involving n gluons, $k + 2$ of negative helicity and $n - k - 2$ of positive helicity.

When is Z_{Gr} well defined?

- Recall that $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ is a linear map, which induces a map $Z_{\mathrm{Gr}} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ on $\mathrm{Gr}_{k,n}^{\geq 0}$. How do we know that Z_{Gr} is well defined on $\mathrm{Gr}_{k,n}^{\geq 0}$, i.e. $\dim(Z_{\mathrm{Gr}}(V)) = k$ for all $V \in \mathrm{Gr}_{k,n}^{\geq 0}$?
- Note: $\dim(Z_{\mathrm{Gr}}(V)) = k \iff Z(v) \neq 0$ for all nonzero $v \in V$.

Lemma

$\bigcup \mathrm{Gr}_{k,n}^{\geq 0} = \{v \in \mathbb{R}^n : \mathrm{var}(v) \leq k-1\}$.

- $\subseteq$ follows from Gantmakher and Krein's theorem. $\supseteq$ is an exercise.

$$(2,0,5,-1,-4,-1,3) \in \begin{bmatrix} 2 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \in \mathrm{Gr}_{3,7}^{\geq 0}$$

When is Z_{Gr} well defined?

Theorem (Karp (2015))

Let $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ *have rank* $k+m$*, and* $W \in \mathrm{Gr}_{k+m,n}$ *be the row span of* Z*. The following are equivalent:*
(i) the map Z_{Gr} *is well defined, i.e.* $\dim(Z_{\mathrm{Gr}}(V)) = k$ *for all* $V \in \mathrm{Gr}_{k,n}^{\geq 0}$*;*
(ii) $\mathrm{var}(v) \geq k$ *for all nonzero* $v \in \ker(Z) = W^{\perp}$*; and*
(iii) $\overline{\mathrm{var}}((\Delta_{J\setminus\{i\}}(W))_{i\in J}) \leq m$ *for all* $J \in \binom{[n]}{k+m+1}$ *with* $\dim(W_J) = k+m$*.*

- e.g. Let $Z := \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 3 \end{bmatrix}$, so $n = 4$, $k+m = 2$. The 4 relevant sequences of Plücker coordinates (as J ranges over $\binom{[4]}{3}$)) are

$(\Delta_{\{2,3\}}, \Delta_{\{1,3\}}, \Delta_{\{1,2\}}) = (-1,-3,5), (\Delta_{\{3,4\}}, \Delta_{\{1,4\}}, \Delta_{\{1,3\}}) = (4,5,-3),$

$(\Delta_{\{2,4\}}, \Delta_{\{1,4\}}, \Delta_{\{1,2\}}) = (-5,5,5), \quad (\Delta_{\{3,4\}}, \Delta_{\{2,4\}}, \Delta_{\{2,3\}}) = (4,-5,-1).$

The maximum number of sign changes among these 4 sequences is 1, which is at most $2-k$ iff $k \leq 1$. Hence Z_{Gr} is well defined iff $k \leq 1$.

When is Z_{Gr} well defined?

Theorem (Karp (2015))

Let $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ have rank $k+m$, and $W \in \mathrm{Gr}_{k+m,n}$ be the row span of Z. The following are equivalent:
(i) the map Z_{Gr} is well defined, i.e. $\dim(Z_{\mathrm{Gr}}(V)) = k$ for all $V \in \mathrm{Gr}_{k,n}^{\geq 0}$;
(ii) $\mathrm{var}(v) \geq k$ for all nonzero $v \in \ker(Z) = W^{\perp}$; and
(iii) $\overline{\mathrm{var}}((\Delta_{J \setminus \{i\}}(W))_{i \in J}) \leq m$ for all $J \in \binom{[n]}{k+m+1}$ with $\dim(W_J) = k+m$.

- If $m = 0$, then (ii) $\Leftrightarrow$ (iii) is a 'dual version' of Gantmakher and Krein's theorem: $V \in \mathrm{Gr}_{k,n}$ is totally positive iff $\mathrm{var}(v) \geq k$ for all $v \in V^{\perp} \setminus \{0\}$.
- Arkani-Hamed and Trnka's condition on Z (for Z to define an amplituhedron) is that its $(k+m) \times (k+m)$ minors are all positive. In this case, Z_{Gr} is well defined by either (ii) or (iii).
- Lam's condition on Z (for Z to define a Grassmann polytope) is that W has a totally positive k-dimensional subspace. This is sufficient by (ii).
- Open problem: is Lam's condition also necessary?

Further directions

- Is there an efficient way to test whether a given $V \in \mathrm{Gr}_{k,n}$ is totally positive using the data of sign patterns? (For Plücker coordinates, in order to test whether V is totally positive, we only need to check that some particular $k(n-k)$ Plücker coordinates are positive, not all $\binom{n}{k}$.)
- Is there a simple way to index the cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ using the data of sign patterns?
- Is there a nice stratification of the subset of the Grassmannian

$$\{V \in \mathrm{Gr}_{k,n} : \mathrm{var}(x) \leq k-1+s \text{ for all } x \in V\},$$

for fixed s? (If $s = 0$, this is $\mathrm{Gr}_{k,n}^{\geq 0}$.)
- I determined when Z_{Gr} is well defined on the totally positive Grassmannian $\mathrm{Gr}_{k,n}^{>0}$. When is Z_{Gr} well defined on a given cell of $\mathrm{Gr}_{k,n}^{\geq 0}$?

Thank you!