# Topology of positive spaces

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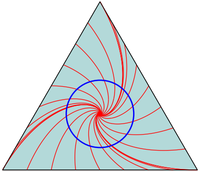
Slides available at www-personal.umich.edu/~snkarp



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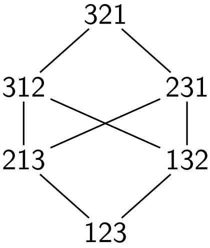
# Introduction to (total) positivity

- A matrix is *totally positive* if every submatrix has a positive determinant.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 71.5987\cdots \\ \lambda_2 &= 3.6199\cdots \\ \lambda_3 &= 0.7168\cdots \\ \lambda_4 &= 0.0646\cdots \end{aligned}$$

- Gantmakher, Krein (1937): The eigenvalues of a square totally positive matrix are all real, positive, and distinct.
- Schoenberg (1930): A totally positive matrix $A$ (regarded as a linear transformation) diminishes sign variation, i.e. $\mathrm{var}(A(x)) \le \mathrm{var}(x)$ for all $x$.
- Other instances of positive spaces:
  - Kellogg (1918): totally positive kernels $K(x, y)$, e.g. $e^{xy}$;
  - Krein (1951): T-systems $(f_1(x), \cdots, f_k(x))$, e.g. $(1, x, x^2, \cdots, x^{k-1})$;
  - Lusztig (1994): $G_{>0}$ and $(G/P)_{>0}$ for algebraic groups $G$, e.g. $\mathrm{SL}_n$;
  - Fomin, Zelevinsky (2002): positive part of a cluster variety, e.g. $\mathrm{Gr}_{k,n}$;
  - Arkani-Hamed, Bai, Lam (2017): positive geometries, e.g. polytopes.

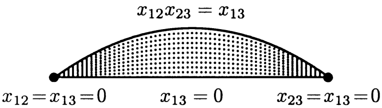
# Topology of positive spaces

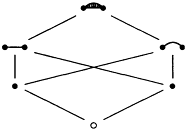



$\mathfrak{S}_3$

$$Y_3 := \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : \begin{array}{l} x + z = 1, \\ \text{all minors} \geq 0 \end{array} \right\}$$

$\leftrightsquigarrow$

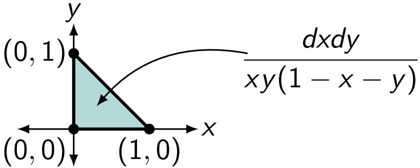


- Edelman (1981) showed that intervals in $\mathfrak{S}_n$ are *shellable*. By results of Björner (1984) and Danaraj and Klee (1974), this implies $\mathfrak{S}_n$ is the face poset of a *regular CW complex* homeomorphic to a ball, the 'next best thing' to a convex polytope.
- Bernstein: Is there a 'naturally occurring' such regular CW complex?
- Fomin and Shapiro (2000) conjectured that $Y_n \subseteq \mathrm{SL}_n$ is such a regular CW complex. This was proved by Hersh (2014), in general Lie type.

# Positive spaces and physics

- Arkani-Hamed, Bai, Lam (2017): a *positive geometry* is a space equipped with a canonical differential form, which has logarithmic singularities at the boundaries of the space. Examples include convex polytopes.



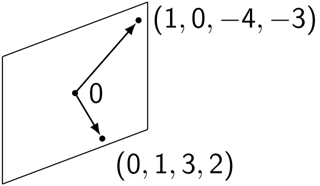

- It appears that the canonical differential forms of several positive geometries have physical significance. So far, three examples are known:

1. *amplituhedra* $\longleftrightarrow$ scattering amplitudes in planar $\mathcal{N}=4$ SYM (Arkani-Hamed, Trnka (2014));
2. associahedra $\longleftrightarrow$ scattering amplitudes in bi-adjoint scalar theories (Arkani-Hamed, Bai, He, Yan (2017+));
3. *cosmological polytopes* $\longleftrightarrow$ wavefunction of the universe in toy models (Arkani-Hamed, Benincasa, Postnikov (2017)).

# The Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{R})$

- The *Grassmannian* $\mathrm{Gr}_{k,n}(\mathbb{R})$ is the set of $k$-dimensional subspaces of $\mathbb{R}^n$.

$V :=$




$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

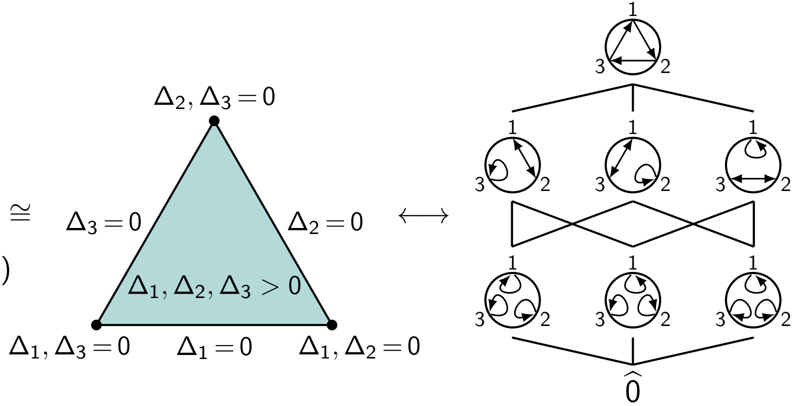
$$\Delta_{\{1,2\}} = 1,\ \Delta_{\{1,3\}} = 3,\ \Delta_{\{1,4\}} = 2,\ \Delta_{\{2,3\}} = 4,\ \Delta_{\{2,4\}} = 3,\ \Delta_{\{3,4\}} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$ in the form of a $k \times n$ matrix, for $k$-subsets $I$ of $\{1, \cdots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of $V$ in columns $I$. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$ *totally nonnegative* if $\Delta_I(V) \geq 0$ for all $k$-subsets $I$. The set of all such $V$ forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.
- Gantmakher, Krein (1950): The element $V$ is totally nonnegative if and only if $\mathrm{var}(x) \leq k - 1$ for all vectors $x$ in $V$.

# The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition due to Rietsch (1998) and Postnikov (2007). Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$ $(= \mathbb{P}^2_{\geq 0})$




- Postnikov showed that the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is given by *circular Bruhat order* on *decorated permutations* with $k$ *anti-excedances*.

# The topology of $\mathrm{Gr}_{k,n}^{\geq 0}$

**Conjecture (Postnikov (2007))**

$\mathrm{Gr}_{k,n}^{\geq 0}$ *and its cell decomposition form a regular CW complex homeomorphic to a ball.*

- Williams (2007): The poset of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ is shellable. In particular, it is the face poset of *some* regular CW complex homeomorphic to a ball.
- Postnikov, Speyer, Williams (2009): $\mathrm{Gr}_{k,n}^{\geq 0}$ is a CW complex (via the *matching polytope* of a plabic graph).
- Rietsch, Williams (2010): $\mathrm{Gr}_{k,n}^{\geq 0}$ is a regular CW complex up to homotopy (via discrete Morse theory).

**Theorem (Galashin, Karp, Lam (2017))**

$\mathrm{Gr}_{k,n}^{\geq 0}$ *is homeomorphic to a closed ball of dimension* $k(n-k)$.

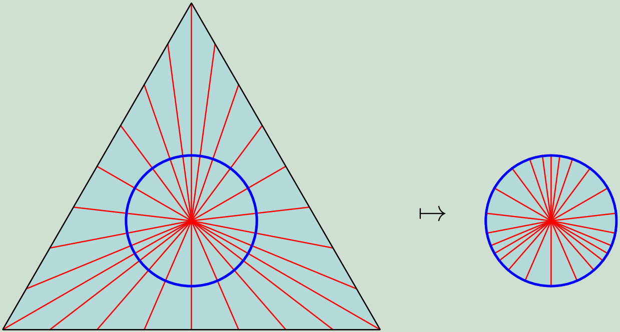
- In order to prove Postnikov's conjecture, one would need to show that the closure of *every* cell of $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a closed ball.

# Showing a compact, convex set is homeomorphic to a ball

**Theorem**

*Every compact, convex subset of $\mathbb{R}^d$ is homeomorphic to a closed ball.*

**Proof**



- How do we generalize this argument to $\mathrm{Gr}_{k,n}^{\geq 0}$?

# Cyclic symmetry of $\mathrm{Gr}_{k,n}^{\geq 0}$

- Define the *(left) cyclic shift map* $\sigma \in \mathrm{GL}_n(\mathbb{C})$ by

$$\sigma(v) := (v_2, v_3, \cdots, v_n, (-1)^{k-1} v_1) \quad \text{for } v = (v_1, \cdots, v_n) \in \mathbb{C}^n.$$

Then $\sigma$ acts on $\mathrm{Gr}_{k,n}(\mathbb{C})$ as an automorphism of order $n$. This cyclic action preserves $\mathrm{Gr}_{k,n}^{\geq 0}$, and gives the 'cyclic symmetry' of the cell decomposition.

$$\begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \overset{\sigma}{\longmapsto} \begin{bmatrix} 1 & -1 & -1 & -2 \\ 1 & 3 & 1 & 0 \end{bmatrix}$$

- What are the fixed points of $\sigma$?

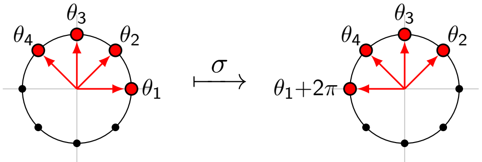
## Theorem (Karp (2017+))

*There are precisely $\binom{n}{k}$ fixed points of $\sigma$ on $\mathrm{Gr}_{k,n}(\mathbb{C})$, namely, the subspaces spanned by eigenvectors $(1, \zeta, \zeta^2, \cdots, \zeta^{n-1})$ of $\sigma$ for some $k$ distinct nth roots $\zeta$ of $(-1)^{k-1}$. There is a unique fixed point $V_0$ in $\mathrm{Gr}_{k,n}^{\geq 0}$, obtained by taking the $k$ eigenvalues $\zeta$ closest to 1 on the unit circle.*

- The Plücker coordinates of $V_0$ are $\Delta_I(V_0) = \prod_{r,s \in I,\ r<s} \sin\left(\frac{s-r}{n}\pi\right)$.

# Moment curves and (bi)cyclic polytopes

- $V_0 \in \mathrm{Gr}_{k,n}^{\geq 0}$ has an elegant description as a vector configuration:
  1. If $k$ is odd, $V_0$ is represented by taking $n$ points on the curve
  $$\left(1, \cos(\theta), \sin(\theta), \cos(2\theta), \sin(2\theta), \cdots, \cos\left(\tfrac{k-1}{2}\theta\right), \sin\left(\tfrac{k-1}{2}\theta\right)\right)$$
  in $\mathbb{R}^k$, equally spaced modulo $2\pi$. If we delete the first component of this curve we obtain the *trigonometric moment curve* in $\mathbb{R}^{k-1}$, which Carathéodory (1911) used to define *cyclic polytopes*.
  2. If $k$ is even, we instead use the *symmetric moment curve*
  $$\left(\cos\left(\tfrac{1}{2}\theta\right), \sin\left(\tfrac{1}{2}\theta\right), \cos\left(\tfrac{3}{2}\theta\right), \sin\left(\tfrac{3}{2}\theta\right), \cdots, \cos\left(\tfrac{k-1}{2}\theta\right), \sin\left(\tfrac{k-1}{2}\theta\right)\right),$$
  which Barvinok and Novik (2007) used to define *bicyclic polytopes*.
- e.g. $k = 2$, $n = 4$:




- These curves also appear as extremal solutions to the *isoperimetric problem for convex curves*, studied by Schoenberg, Krein, and Nudel'man.

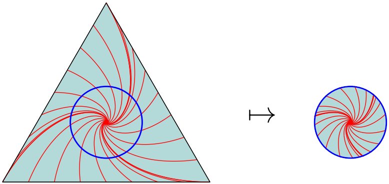
# Showing $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a ball

- Let us now regard $\sigma$ as a vector field on $\mathrm{Gr}_{k,n}(\mathbb{R})$, which sends each $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$ along the trajectory $\exp(t\sigma)(V)$ for $t \geq 0$.

**Lemma (Galashin, Karp, Lam (2017))**

*(i) If $V \in \mathrm{Gr}_{k,n}^{\geq 0}$, then $\exp(t\sigma)(V) \in \mathrm{int}(\mathrm{Gr}_{k,n}^{\geq 0})$ for all $t > 0$.*
*(ii) For all $V \in \mathrm{Gr}_{k,n}^{\geq 0}$, we have $\lim_{t\to\infty} \exp(t\sigma)(V) = V_0$.*

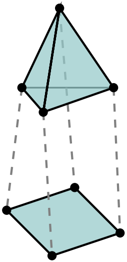
- This partitions $\mathrm{Gr}_{k,n}^{\geq 0} \setminus \{V_0\}$ into trajectories from the boundary to $V_0$.
- e.g. $\mathrm{Gr}_{1,3}^{\geq 0}$



- We use a similar argument to show that the following spaces are closed balls: Fomin and Shapiro's $Y_n$, Lam's compactification of the space of electrical networks, and the *cyclically symmetric amplituhedron*.

# Amplituhedra and Grassmann polytopes

- By definition, a polytope is the image of a simplex under an affine map:



simplex in $\mathbb{P}^{n-1}$ - - - - - - - - - → $\mathrm{Gr}_{k,n}^{\geq 0}$

linear map $\mathbb{R}^n \to \mathbb{R}^{m+1}$ ↓ linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$ ↓

polytope in $\mathbb{P}^m$ - - - - - - - - - → *Grassmann polytope* in $\mathrm{Gr}_{k,k+m}$

A *Grassmann polytope* is the image of a map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

- When $Z$ has positive maximal minors, the Grassmann polytope is called an *amplituhedron*. Amplituhedra generalize cyclic polytopes into the Grassmannian. They were introduced by Arkani-Hamed and Trnka (2014), and inspired Lam (2015) to define Grassmann polytopes.
- When $Z$ represents the point $V_0 \in \mathrm{Gr}_{k+m,n}^{\geq 0}$, we show the resulting *cyclically symmetric amplituhedron* is homeomorphic to a ball.

# Open questions

- Can we adapt our argument to show that other positive spaces are homeomorphic to balls? What about the closures of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, or arbitrary amplituhedra? What can we say about the topology of Grassmann polytopes? (Note: they are not all closed balls!)
- In studying mirror symmetry for partial flag varieties, Rietsch (2008) introduced the *superpotential*, a rational function on $\mathrm{Gr}_{k,n}$:

$$\left( \sum_{1 \le i \le n,\ i \neq n-k} \frac{\Delta_{\{i+1, i+2, \cdots, i+k-1, i+k+1\}}}{\Delta_{\{i+1, i+2, \cdots, i+k\}}} \right) + q \frac{\Delta_{\{n-k+1, n-k+2, \cdots, n-1, 1\}}}{\Delta_{\{n-k+1, n-k+2, \cdots, n\}}}.$$

It turns out that the critical points of the superpotential (for fixed $q$) are precisely the fixed points of the *$q$-deformed cyclic shift map* $\sigma_q$. How exactly is the superpotential related to the cyclic shift map? Can we use its gradient flow to show that $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a ball?

# Thank you!