$q$-Whittaker functions, finite fields, and Jordan forms

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arXiv:2110.02301

September 30th, 2022
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Schur functions

- A partition $\lambda$ is a weakly-decreasing sequence of nonnegative integers.

- e.g. $\lambda = (4, 4, 1)$

- A semistandard tableau $T$ is a filling of $\lambda$ with positive integers which is weakly increasing across rows and strictly increasing down columns.

Definition (Schur function)

$$s_{\lambda}(x_1, x_2, \ldots) := \sum_{T} x^T,$$

where the sum is over all semistandard tableaux $T$ of shape $\lambda$.

- $s_{\lambda}(x)$ is symmetric in the variables $x_i$. 
Schur functions

e.g. \( s_{(2,1)}(x_1, x_2, x_3) = \)

\[
x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2
\]

Schur functions appear in many contexts; for example, they:

- form an *orthonormal basis* for the algebra of symmetric functions in \( x \);
- are characters of the *irreducible polynomial representations* of \( \text{GL}_n(\mathbb{C}) \);
- give the values of the *irreducible characters* of the symmetric group \( S_n \), when expanded in terms of power sum symmetric functions;
- are representatives for *Schubert classes* in the cohomology ring of the Grassmannian \( \text{Gr}_{k,n}(\mathbb{C}) \);
- define the *Schur processes* of Okounkov and Reshetikhin (2003).
Cauchy identity

Theorem (Cauchy)

\[
\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x)s_\lambda(y)
\]

- The identity is equivalent to the orthonormality of the Schur functions. It also gives the partition function for the Schur processes.
- The left-hand side counts \textit{nonnegative-integer matrices}, and the right-hand side counts \textit{pairs of semistandard tableaux of the same shape}.
- e.g. Taking the coefficient of \(x_1 x_2 y_1 y_2\) on each side gives

\[
1 + 1 = 1 + 1
\]

\[
12, 21 \quad \left(\frac{1}{2}, \frac{1}{2}\right) \quad \left(1 2, 1 2\right)
\]
The *Burge correspondence* (also known as *column Robinson–Schensted–Knuth*) is a bijection

\[ M \mapsto (P(M), Q(M)) \]

between nonnegative-integer matrices and pairs of semistandard tableaux of the same shape. It proves the Cauchy identity for Schur functions.

- \( P(M) \) is obtained via *column insertion* and \( Q(M) \) via *recording*.
- e.g. \( w = 25143 \)

\[
\begin{array}{cccc}
2 & 2 & 1 & 2 \\
5 & 5 & 4 & 5 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 5 \\
3 & 4 \\
2 & 4 \\
\end{array}
\]

\( P(w) \)

\( Q(w) \)
Nilpotent matrices

- An $n \times n$ matrix $N$ over $\mathbb{k}$ is nilpotent if some power of $N$ is zero. Such an $N$ can be conjugated over $\mathbb{k}$ into Jordan form. Let $JF^\top(N)$ be the transpose of the partition given by the sizes of the Jordan blocks.

  e.g. \[
  \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0
  \end{bmatrix} \mapsto \begin{bmatrix}
\end{bmatrix} \mapsto \begin{bmatrix}
\end{bmatrix} \mapsto \begin{bmatrix}
\end{bmatrix}
\]

- Algebraically, $JF^\top(N)$ is the partition $\lambda$ given by

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_i = \dim(\ker(N^i)) \quad \text{for all } i.
\]

Theorem (Gansner (1981))

Let $N$ be a generic $n \times n$ strictly upper-triangular matrix, where $N_{i,j} = 0$ for all inversions $(i,j)$ of $w^{-1}$. Then $P(w)$ and $Q(w)$ can be read off from the Jordan forms of the leading submatrices of $N$ and $w^{-1}Nw$. 
Burge correspondence via Jordan forms

\[ e.g. \ w = 25143 \]

\[ N = \begin{bmatrix}
0 & 0 & a & b & 0 \\
0 & 0 & c & d & e \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad (a, b, c, d, e \in k \text{ generic}) \]

\[ P(w): \]

\[ Q(w): \]
Flag variety

- A complete flag $F$ in $\mathbb{k}^n$ is a sequence of nested subspaces
  
  \[ 0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = \mathbb{k}^n, \quad \dim(F_i) = i \text{ for all } i. \]

- An $n \times n$ (nilpotent) matrix $N$ is strictly compatible with $F$ if
  
  \[ N(F_i) \subseteq F_{i-1} \quad \text{for all } i. \]

- The matrix $N$ in Gansner’s theorem is precisely one which is strictly compatible with two complete flags $F$ and $F'$ defined by
  
  \[ F_i := \langle e_1, e_2, \ldots, e_i \rangle \quad \text{and} \quad F'_j := \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(j)} \rangle. \]

  The two sequences of matrices in the theorem are $(N|_{F_i})_{i=1}^n$ and $(N|_{F'_j})_{j=1}^n$.

- More generally, we can take any pair of flags $(F, F')$ with relative position $w$, denoted $F \xrightarrow{w} F'$. The relative position records $\dim(F_i \cap F'_j)$ for all $i$ and $j$, or alternatively, the Schubert cell of $F'$ relative to $F$. 
Theorem (Steinberg (1976, 1988), Spaltenstein (1982), Rosso (2012))

Fix partial flags $F$ and $F'$ with $F \xrightarrow{M} F'$. Let $N$ be a generic nilpotent matrix strictly compatible with both $F$ and $F'$. Then

$$P(M) = JF^\top(N; F) \quad \text{and} \quad Q(M) = JF^\top(N; F').$$

- If $F \xrightarrow{w} F'$, then $F' \xrightarrow{w^{-1}} F$. This implies the symmetry
  $$P(w^{-1}) = Q(w).$$

- What happens when $\mathbb{k}$ is a finite field, and we consider all choices of $N$ (not necessarily generic)?
**q-Whittaker functions**

- Define \([n]_q := 1 + q + q^2 + \cdots + q^{n-1}\) and \([n]_q! := [n]_q[n-1]_q \cdots [1]_q\).

**Definition (q-Whittaker function)**

\[
W_\lambda(x_1, x_2, \ldots; q) := \sum_T \mathrm{wt}_q(T) x^T,
\]

where the sum is over all semistandard tableaux \(T\) of shape \(\lambda\).

- \(W_\lambda(x; q)\) is symmetric in the variables \(x_i\), and specializes to \(s_\lambda(x)\) when \(q = 0\). We obtain the \(\mathfrak{gl}_n\)-Whittaker functions as a certain \(q \to 1\) limit.

- e.g. \(T = \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 7 \\
6 & & \\
\end{array}\)
  \[
  \mathrm{wt}_q(T) = [1]_q[2]_q[1]_q[2]_q[2]_q[1]_q[2]_q = (1 + q)^4
  \]

- We have the following specializations:

  \[
  W_\lambda(x; q) = P_\lambda(x; q, 0) = q^{\deg(\tilde{H}_\lambda)} \omega(\tilde{H}_\lambda(x; 1/q, 0)),
  \quad W_\lambda(x; 1) = e_\lambda^\top(x).
  \]
Theorem (Macdonald (1995))

\[
\prod_{i, j \geq 1} \prod_{d \geq 0} \frac{1}{1 - x_i y_j q^d} = \sum_{\lambda} \frac{(1 - q)^{-\lambda_1}}{\prod_{i \geq 1} [\lambda_i - \lambda_{i+1}] q!} \mathcal{W}_\lambda(x; q) \mathcal{W}_\lambda(y; q)
\]

This gives the partition function for the \textit{q-Whittaker processes}, a special case of the \textit{Macdonald processes} of Borodin and Corwin (2014).

e.g. Taking the coefficient of \(x_1 x_2 y_1 y_2\) on each side gives

\[
(1 - q)^{-2} + (1 - q)^{-2} = (1 - q)^{-1} + (1 - q)^{-2}(1 + q)
\]

\[
\begin{pmatrix} 12 \end{pmatrix} \quad \begin{pmatrix} 1 \ 2 \end{pmatrix} \quad \begin{pmatrix} 1 \ 2 \end{pmatrix}
\]

\[
1 - q \quad 1 \quad q
\]
$q$-Burge correspondence

- e.g. $w = 12 \quad N = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad (a \in \mathbb{F}_{1/q})$

$$P(w): \quad 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad 1 \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad Q(w): \quad 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad 1 \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

- $a \neq 0$: \[ \begin{array}{c} \square \\ \frac{1}{2} \end{array} \quad \square \quad \frac{1}{2} \quad \mathbb{P} = 1 - q \]
- $a = 0$: \[ \begin{array}{c} \square \\ 1 \, 2 \end{array} \quad \square \quad 1 \, 2 \quad \mathbb{P} = q \]

- e.g. $w = 21 \quad N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$P(w): \quad 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad 1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad Q(w): \quad 2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad 2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- $P = 1$
Let $1/q$ be a prime power, and fix partial flags $F \xrightarrow{M} F'$ over $\mathbb{F}_{1/q}$. For semistandard tableaux $T$ and $T'$ of the same shape, define

$$p_M(T, T') := \mathbb{P}(JF^T(N; F) = T \text{ and } JF^T(N; F') = T'),$$

where $N$ is a uniformly random nilpotent matrix strictly compatible with both $F$ and $F'$. (This does not depend on the choice of $(F, F')$.)

**Theorem (Karp, Thomas (2022))**

(i) The maps $p_M(\cdot, \cdot)$ define a probabilistic bijection proving the Cauchy identity for $q$-Whittaker functions, called the $q$-Burge correspondence.

(ii) The bijection converges to the classical Burge correspondence as $q \to 0$.

The inverse probabilities are also given by $(\ast)$, but where $N$ is fixed and $(F, F')$ is uniformly random.

Two other probabilistic bijections were given by Matveev and Petrov (2017), using $q$-analogues of row and column insertion.
Proof outline

**Theorem (Borho, MacPherson (1983); Karp, Thomas (2022))**

Fix a nilpotent matrix $N$ over $\mathbb{F}_{1/q}$ with Jordan type $\lambda$. The coefficient of $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ in $W_\lambda(x; q)$ equals $q \sum_i (\binom{\lambda_i}{2} - \binom{\alpha_i}{2})$ times the number of partial flags $F$ over $\mathbb{F}_{1/q}$ strictly compatible with $N$ satisfying

$$\dim(F_i) = \alpha_1 + \cdots + \alpha_i \quad \text{for all } i.$$ 

- e.g. $\lambda = \begin{array}{c|c|c|c} & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \end{array}$, $N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then the coefficient of $x_1x_2$ in $W_\lambda(x; q)$ is $q^1 \cdot \#(\text{complete flags in } \mathbb{F}^2_{1/q}) = q(1 + 1/q) = q + 1$.

- This is similar to a formula for the modified Hall–Littlewood functions $\tilde{H}_\lambda(x; q, 0)$ in terms of weakly compatible flags over $\mathbb{F}_q$.

- A key step to proving both theorems is enumerating an arbitrary double coset of $P_\alpha \backslash \text{GL}_n(\mathbb{F}_{1/q}) / P_\beta$, where $P_\alpha$ and $P_\beta$ are standard parabolic subgroups of $\text{GL}_n(\mathbb{F}_{1/q})$. 
Combinatorics of the $q$-Burge correspondence

Problem

Is $p_M(T, T')$ a rational function of $q$? (If so, it is a polynomial.)

- We have an explicit formula when $M$ is a diagonal matrix (i.e. $F = F'$).

Problem

Is there a recursive combinatorial rule for calculating $p_M(T, T')$?

- Unlike insertion-based deformations of RSK, the $q$-Burge correspondence does not admit Fomin-style local growth rules. For example, the diagram

\[
\begin{bmatrix}
\begin{array}{c}
F'_{j-1} \\
F'_{j}
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
F_{i-1} \\
F_i
\end{array}
\end{bmatrix}
\]

with $M_{i,i} = 0$ can be completed to either \[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\] or \[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\].
Quiver representations and the preprojective algebra

- Consider a path quiver with a unique sink:

\[ Q = \begin{array}{c}
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet \\
\end{array} \]

- A representation \( V \) of \( Q \) is an assignment of a vector space to each vertex and a linear map to each arrow, e.g.,

\[ V = \begin{array}{c}
\begin{array}{c}
\mathbb{k} & [-1] & \mathbb{k} & [3] & \mathbb{k}^2 & [2 & 1] & \mathbb{k}^2 & 0 & 0 \\
\end{array}
\end{array} \]

- We will only consider \( V \) where every linear map is injective. Isomorphism classes of such \( V \) are indexed by nonnegative-integer matrices \( M \).

- We now decorate \( V \) with a linear map for the reverse of each arrow, such that a relation holds for every vertex:

\[ \alpha \circ \gamma \pm \delta \circ \beta = 0 \]

This defines a module \( V^\# \) over the preprojective algebra of \( Q \).
Up to isomorphism, $V^\#$ is given (non-uniquely) by a triple $(F, F', N)$:

$$V^\# = \begin{array}{cccccc}
F_1 & \text{id} & F_2 & \text{id} & F_3 = F'_3 & \text{id} & F'_2 & \text{id} & F'_1 \\
N & \quad & N & \quad & -N & \quad & -N
\end{array}$$

The socle filtration of $V^\#$ corresponds precisely to the pair of tableaux

$$(T, T') = (JF^T(N; F), JF^T(N; F')).$$

e.g.

$$V^\# = \begin{array}{cccccc}
\langle e_1 \rangle & \text{id} & k^3 & \text{id} & \langle e_1 \rangle \\
\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{array} \quad \iff \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$
Counting isomorphism classes

- The $q$-Burge correspondence implies enumerative results about such modules $V^\#$. For example:

**Theorem (Karp, Thomas (2022))**

Let $(T, T')$ be a pair of semistandard tableaux of shape $\lambda$, and let $d$ be a dimension vector of $Q$. Then

$$
\sum_{[V^\#]} \frac{1}{|\text{Aut}(V^\#)|} = \frac{q^{c(d)}(1 - q)^{-\lambda_1}}{\prod_{i \geq 1} [\lambda_i - \lambda_{i+1}] q!} \text{wt}_q(T) \text{wt}_q(T'),
$$


where the sum is over all isomorphism classes $[V^\#]$ of modules $V^\#$ over $\mathbb{F}_1/q$ with dimension vector $d$ and socle filtration corresponding to $(T, T')$.

**Thank you!**