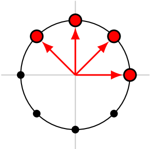
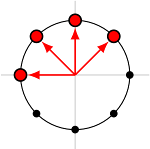
Cyclic symmetry in the Grassmannian

Slides available at `www-personal.umich.edu/~snkarp`



$\longleftrightarrow$



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including joint work with Pavel Galashin and Thomas Lam

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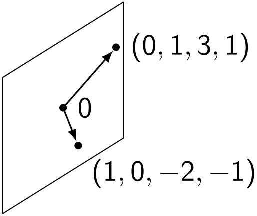
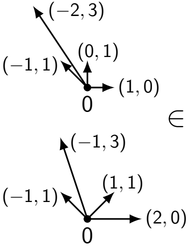
University of Minnesota, Twin Cities

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces V of $\mathbb{C}^n$.

$V :=$ 

$$= \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 3 & 1 \end{bmatrix} =$$



$\in \mathrm{Gr}_{2,4}^{\geq 0}$

$$\Delta_{\{1,2\}} = 1,\ \Delta_{\{1,3\}} = 3,\ \Delta_{\{1,4\}} = 1,\ \Delta_{\{2,3\}} = 2,\ \Delta_{\{2,4\}} = 1,\ \Delta_{\{3,4\}} = 1$$

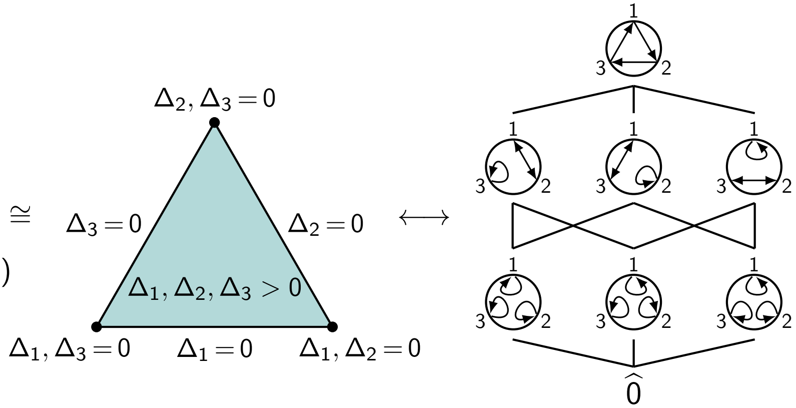
- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \ldots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call $V \in \mathrm{Gr}_{k,n}$ *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all such V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.
- We can also view an element $V \in \mathrm{Gr}_{k,n}$ as a vector configuration of n ordered vectors in $\mathbb{C}^k$ (up to automorphism).

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition due to Rietsch (1998) and Postnikov (2007). Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$

$(= \mathbb{P}^2_{\geq 0})$



- Postnikov showed that the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is given by *circular Bruhat order* on *decorated permutations* with k *anti-excedances*.

Cyclic symmetry of $\mathrm{Gr}_{k,n}^{\geq 0}$

- Define the *(left) cyclic shift map* $\sigma \in \mathrm{GL}_n(\mathbb{C})$ by

$$\sigma(v) := (v_2, v_3, \cdots, v_n, (-1)^{k-1} v_1) \quad \text{for } v = (v_1, \cdots, v_n) \in \mathbb{C}^n.$$

Then σ acts on $\mathrm{Gr}_{k,n}$ as an automorphism of order n. This cyclic action preserves $\mathrm{Gr}_{k,n}^{\geq 0}$, and gives the 'cyclic symmetry' of the cell decomposition.

$$\begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \xmapsto{\sigma} \begin{bmatrix} 1 & -1 & -1 & -2 \\ 1 & 3 & 1 & 0 \end{bmatrix}$$

- What are the fixed points of σ? They are the k-dimensional invariant subspaces of $\mathbb{C}^n$, spanned by k linearly independent eigenvectors of σ.

Theorem (Karp (2017+))

There are precisely $\binom{n}{k}$ fixed points of σ on $\mathrm{Gr}_{k,n}$, namely, the subspaces spanned by eigenvectors $(1, \zeta, \zeta^2, \cdots, \zeta^{n-1})$ for some k distinct nth roots ζ of $(-1)^{k-1}$. There is a unique totally nonnegative fixed point V_0, obtained by taking the k eigenvalues ζ closest to 1 on the unit circle.

Fixed points of the cyclic shift map

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These values of ζ are $e^{-i\frac{(k-1)\pi}{n}}, e^{-i\frac{(k-3)\pi}{n}}, \cdots, e^{i\frac{(k-3)\pi}{n}}, e^{i\frac{(k-1)\pi}{n}}$.

Proof

1. Scott (1879): $\Delta_I(V_0) = \displaystyle\prod_{r,s\in I,\ r<s} \sin\left(\frac{s-r}{n}\pi\right)$ for all k-subsets I.
2. Gantmakher, Krein (1950): V is in $\mathrm{Gr}_{k,n}^{\geq 0}$ if and only if for all $v \in V$, we have $\mathrm{Re}(v) \in V$ and $\mathrm{Re}(v)$ changes sign at most $k-1$ times. Observe that if $\zeta = e^{i\theta}$, then $\mathrm{Re}(1, \zeta, \zeta^2, \cdots, \zeta^{n-1})$ changes sign at least $\frac{n|\theta|}{\pi} - 1$ times.

Moment curves and (bi)cyclic polytopes

- $V_0 \in \mathrm{Gr}_{k,n}^{\geq 0}$ has an elegant description as a vector configuration. Let $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi$ be equally spaced on the real line.
- If k is odd, V_0 is represented by the points $\theta = \theta_1, \cdots, \theta_n$ on the curve

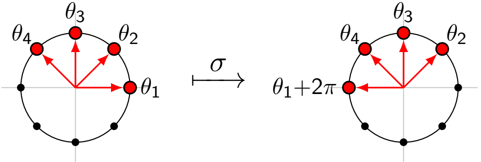
$$\big(1, \cos(\theta), \sin(\theta), \cos(2\theta), \sin(2\theta), \cdots, \cos\big(\tfrac{k-1}{2}\theta\big), \sin\big(\tfrac{k-1}{2}\theta\big)\big) \text{ in } \mathbb{R}^k.$$

If we delete the first component we obtain the *trigonometric moment curve* in $\mathbb{R}^{k-1}$, which Carathéodory (1911) used to define *cyclic polytopes*.

- If k is even, we instead use the *symmetric moment curve*

$$\big(\cos\big(\tfrac{1}{2}\theta\big), \sin\big(\tfrac{1}{2}\theta\big), \cos\big(\tfrac{3}{2}\theta\big), \sin\big(\tfrac{3}{2}\theta\big), \cdots, \cos\big(\tfrac{k-1}{2}\theta\big), \sin\big(\tfrac{k-1}{2}\theta\big)\big) \text{ in } \mathbb{R}^k.$$

Barvinok and Novik (2007) used this curve to define *bicyclic polytopes*.

- e.g. $k = 2$, $n = 4$: 

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- These curves also appear as extremal solutions to the *isoperimetric problem for convex curves*, studied by Schoenberg, Krein, and Nudel'man.

Quantum cohomology of $\mathrm{Gr}_{k,n}$

- The *cohomology ring* $H^*(\mathrm{Gr}_{k,n})$ has dimension $\binom{n}{k}$. A basis is $\{[X_\lambda] : \lambda \subseteq k \times (n-k)\}$, where $X_\lambda \subseteq \mathrm{Gr}_{k,n}$ is the *Schubert variety* indexed by the partition λ. Multiplication is given by

$$[X_\lambda] \cdot [X_\mu] = \sum_\nu c_{\lambda,\nu}^{\nu}[X_\nu],$$

where the $c_{\lambda,\nu}^{\nu}$'s are *Littlewood–Richardson coefficients*. We have

$$H^*(\mathrm{Gr}_{k,n}) \cong \mathbb{C}[e_1, \cdots, e_k]/(h_{n-k+1}, \cdots, h_n), \text{ via } [X_\lambda] \mapsto s_\lambda.$$

- The *quantum cohomology ring* $QH^*(\mathrm{Gr}_{k,n})$ is a q-deformation of $H^*(\mathrm{Gr}_{k,n})$. By definition, it has the basis $\{[X_\lambda] : \lambda \subseteq k \times (n-k)\}$, with multiplication

$$[X_\lambda] *_q [X_\mu] := \sum_{d,\nu} \langle X_\lambda, X_\mu, X_{\nu^\vee}\rangle_d q^d [X_\nu].$$

Here $\langle X_\lambda, X_\mu, X_{\nu^\vee}\rangle_d$ counts rational curves passing through generic translates of X_λ, X_μ, and $X_{\nu^\vee}$. Siebert and Tian (1994) showed that

$$QH^*(\mathrm{Gr}_{k,n}) \cong \mathbb{C}[e_1, \cdots, e_k, q]/(h_{n-k+1}, \cdots, h_{n-1}, h_n - (-1)^{k-1}q).$$

Quantum cohomology of $\mathrm{Gr}_{k,n}$

- In unpublished work, Peterson discovered a presentation of the quantum cohomology ring of a general partial flag variety, as the coordinate ring of a *Peterson variety*. In the case of $\mathrm{Gr}_{k,n}$, this was proved by Rietsch (2001). We can identify the Peterson variety $\mathcal{P}_{k,n}$ with a certain subvariety of $\mathrm{Gr}_{k,n}$.
- For $q \neq 0$, define the *q-deformed cyclic shift map* $\sigma_q \in \mathrm{GL}_n(\mathbb{C})$ by

$$\sigma_q(v) := (v_2, v_3, \cdots, v_n, (-1)^{k-1} q v_1) \quad \text{for } v = (v_1, \cdots, v_n) \in \mathbb{C}^n.$$

Then σ_q acts on $\mathrm{Gr}_{k,n}$ as an automorphism of order n.

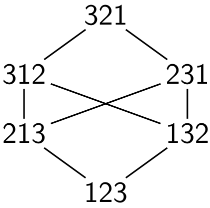
Proposition (Karp (2017+))

The Peterson variety $\mathcal{P}_{k,n} \subseteq \mathrm{Gr}_{k,n}$ is the union of $\mathrm{span}\{e_1, \cdots, e_k\}$ *with the sets of fixed points of σ_t over all $t \neq 0$. Explicitly, the isomorphism $QH^*(\mathrm{Gr}_{k,n}) \to \mathbb{C}[\mathcal{P}_{k,n}]$ sends q to a map $\mathcal{P}_{k,n} \to \mathbb{C}$, whose fiber over $t \neq 0$ equals the set of fixed points of σ_t.*

Topology of positive spaces

Theorem (Galashin, Karp, Lam (2017))

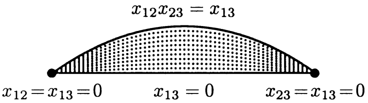
$\mathrm{Gr}_{k,n}^{\geq 0}$ *is homeomorphic to a closed ball of dimension* $k(n-k)$.

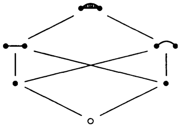


$\mathfrak{S}_3$

$\leftsquigarrow$

$$Y_3 = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : \begin{array}{l} x + z = 1, \\ \text{all minors} \geq 0 \end{array} \right\}$$





- Edelman (1981) showed that intervals in $\mathfrak{S}_n$ are *shellable*. By results of Björner (1984), this implies $\mathfrak{S}_n$ is the face poset of a *regular CW complex* homeomorphic to a ball. Is there a 'naturally occurring' such CW complex?
- Fomin and Shapiro (2000) conjectured that $Y_n \subseteq \mathrm{SL}_n$ is such a regular CW complex. This was proved by Hersh (2014), in general Lie type.

Topology of positive spaces

Theorem (Galashin, Karp, Lam (2017))

$\mathrm{Gr}_{k,n}^{\geq 0}$ *is homeomorphic to a closed ball of dimension* $k(n-k)$.

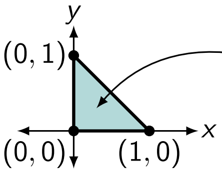
Conjecture (Postnikov (2007))

$\mathrm{Gr}_{k,n}^{\geq 0}$ *and its cell decomposition form a regular CW complex homeomorphic to a ball.*

- Williams (2007): The poset of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ is shellable.
- Postnikov, Speyer, Williams (2009): $\mathrm{Gr}_{k,n}^{\geq 0}$ is a CW complex.
- Rietsch, Williams (2010): $\mathrm{Gr}_{k,n}^{\geq 0}$ is a regular CW complex up to homotopy (via discrete Morse theory).
- In order to prove Postnikov's conjecture, one needs to show that the closure of *every* cell of $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a closed ball.

Positive spaces and physics

- Arkani-Hamed, Bai, Lam (2017): a *positive geometry* is a space equipped with a canonical differential form, which has simple poles (only) at the boundaries of the space. Examples include polytopes and $\mathrm{Gr}_{k,n}^{\geq 0}$.



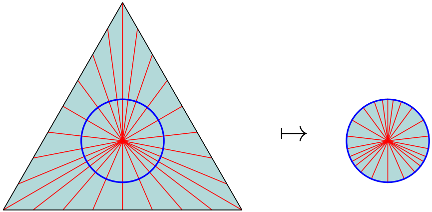
$$\frac{dxdy}{xy(1-x-y)}$$

- It appears that the canonical differential forms of several positive geometries have physical significance. So far, three examples are known:

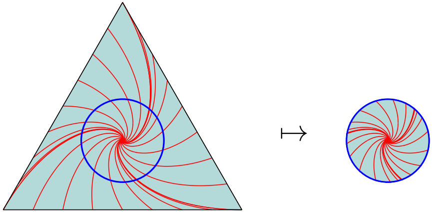
1. *amplituhedra* $\longleftrightarrow$ scattering amplitudes in planar $\mathcal{N} = 4$ SYM (Arkani-Hamed, Trnka (2014))
2. associahedra $\longleftrightarrow$ scattering amplitudes in bi-adjoint scalar theories (Arkani-Hamed, Bai, He, Yan (2017+));
3. *cosmological polytopes* $\longleftrightarrow$ wavefunction of the universe in toy models (Arkani-Hamed, Benincasa, Postnikov (2017))

Showing $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a ball

- How do we show that a compact, convex subset of $\mathbb{R}^d$ is homeomorphic to a ball?



- For $\mathrm{Gr}_{k,n}^{\geq 0}$, we apply the flow given by the vector field σ, which takes V to $\exp(t\sigma)(V)$ in time t. This flow attracts all of $\mathrm{Gr}_{k,n}^{\geq 0}$ toward V_0.
- e.g. $\mathrm{Gr}_{1,3}^{\geq 0}$



Open questions

- Can we adapt the proof to show that other positive spaces are homeomorphic to balls? What about the closures of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$?
- In studying mirror symmetry for partial flag varieties, Rietsch (2008) introduced the *superpotential*, a rational function on $\mathrm{Gr}_{k,n}$:

$$\sum_{i=1}^{n} \frac{\Delta_{\{i+1,i+2,\cdots,i+k-1,i+k+1\}}}{\Delta_{\{i+1,i+2,\cdots,i+k\}}}.$$

It turns out that the critical points of the superpotential are precisely the fixed points of σ. How exactly is the superpotential related to σ? Can we use its gradient flow to show that $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a ball?

Thank you!