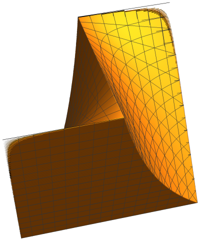
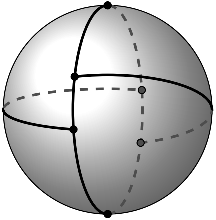
Topology of positive spaces

Slides available at `www-personal.umich.edu/~snkarp`





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joint work with Pavel Galashin and Thomas Lam

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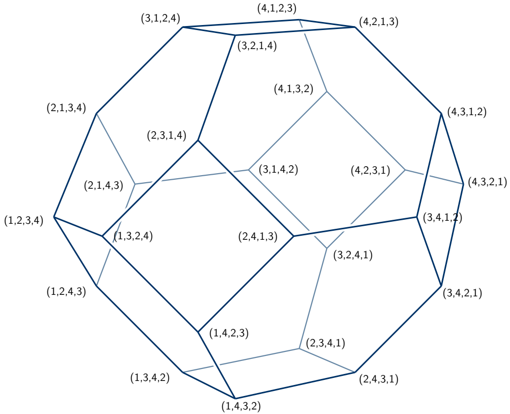
Massachusetts Institute of Technology, Microsoft Research

Permutohedron

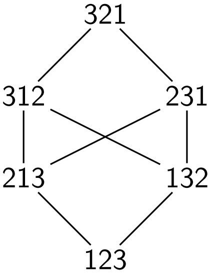


- The vertices of the permutohedron are $(\pi(1), \cdots, \pi(n)) \in \mathbb{R}^n$ for $\pi \in \mathfrak{S}_n$.
- The edges of the permutohedron are

$$(\cdots, i, \cdots, i+1, \cdots) \quad \longleftrightarrow \quad (\cdots, i+1, \cdots, i, \cdots).$$

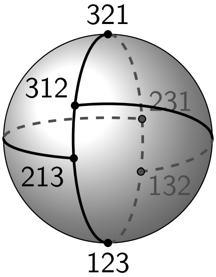
These correspond to cover relations in the *weak Bruhat order* on $\mathfrak{S}_n$.

Permutohedron for the strong Bruhat order?



$\mathfrak{S}_3$ (strong order)

$\rightsquigarrow$



?

- Using *total positivity*, we can define a space whose d-dimensional faces correspond to intervals of length d in the strong Bruhat order on $\mathfrak{S}_n$.
- This space is not a polytope! However, topologically it is just as good:
  1. it is partitioned into faces F, each homeomorphic to an open ball;
  2. the boundary ∂F of each face F is a union of lower-dimensional faces;
  3. the closure $\overline{F}$ of each face F is homeomorphic to a closed ball.

Such a space is called a *regular CW complex*.

Introduction to total positivity

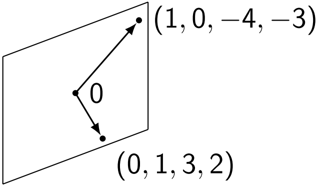
- A matrix is *totally positive* if every submatrix has positive determinant.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 71.5987\cdots \\ \lambda_2 &= 3.6199\cdots \\ \lambda_3 &= 0.7168\cdots \\ \lambda_4 &= 0.0646\cdots \end{aligned}$$

- Gantmakher, Krein (1937): the eigenvalues of a square totally positive matrix are all real, positive, and distinct.
- Totally positive matrices are a discrete analogue of *totally positive kernels* (e.g. $K(x,y) = e^{xy}$), introduced by Kellogg (1918).
- Lusztig (1994): total positivity for algebraic groups G (e.g. $G = \mathrm{SL}_n$) and partial flag varieties G/P (e.g. $G/P = \mathrm{Gr}_{k,n}$, Fl_n).
- Fomin, Zelevinsky (2002): cluster algebras.
- Postnikov (2006): *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$. It has been related to the ASEP, the KP equation, Poisson geometry, quantum matrices, scattering amplitudes, mirror symmetry, singularities of curves, ...

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

$V :=$ 

$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

$$\Delta_{12} = 1, \ \ \Delta_{13} = 3, \ \ \Delta_{14} = 2, \ \ \Delta_{23} = 4, \ \ \Delta_{24} = 3, \ \ \Delta_{34} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \cdots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call $V \in \mathrm{Gr}_{k,n}$ *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all such V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.
- We can think of $\mathrm{Gr}_{k,n}^{\geq 0}$ as a compactification of the space of $k \times (n-k)$ totally positive matrices, or as the Grassmannian notion of a simplex.

Compactifying the space of (totally positive) matrices

- The closure of the space of $k \times \ell$ totally positive matrices is not compact, e.g. consider

$$\begin{bmatrix} 1 & 1 \\ 1 & t \end{bmatrix} \text{ as } t \to \infty.$$

- We can embed the space of $k \times \ell$ matrices inside $\mathrm{Gr}_{k,k+\ell}$ as the subset where the Plücker coordinate $\Delta_{\{1,2,\cdots,k\}}$ is nonzero:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & g & h & i \\ 0 & 1 & 0 & -d & -e & -f \\ 0 & 0 & 1 & a & b & c \end{bmatrix} \in \mathrm{Gr}_{3,3+3}\,.$$

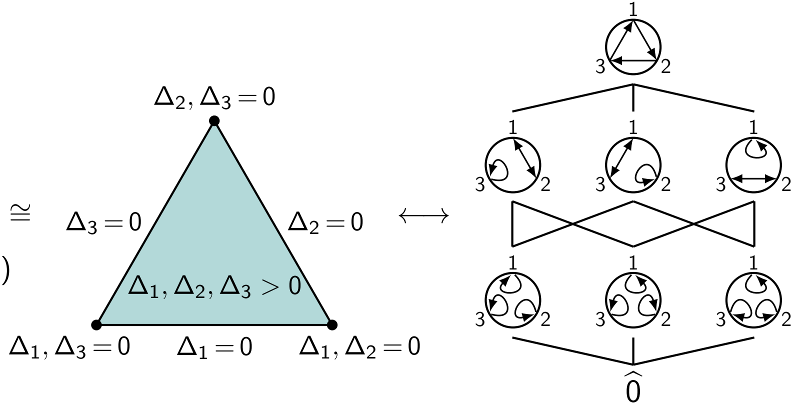
- This identifies the space of $k \times \ell$ totally positive matrices with the totally positive part of $\mathrm{Gr}_{k,k+\ell}$, whose closure is the compact space $\mathrm{Gr}_{k,k+\ell}^{\geq 0}$.
- Passing to the Grassmannian also reveals certain hidden symmetries, notably the *cyclic symmetry* given by the action

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \mapsto \begin{bmatrix} | & & | & | \\ v_2 & \cdots & v_n & (-1)^{k-1}v_1 \\ | & & | & | \end{bmatrix} \quad \text{on } \mathrm{Gr}_{k,n}^{\geq 0}\,.$$

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition due to Rietsch (1998) and Postnikov (2006). Each *positroid cell* is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$ $(= \mathbb{P}_{\geq 0}^{2})$

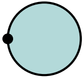
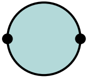
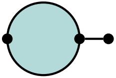


- Postnikov showed that the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is given by *circular Bruhat order* on *decorated permutations* with k *anti-excedances*.

The topology of $\mathrm{Gr}_{k,n}^{\geq 0}$

Conjecture (Postnikov (2006))

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ is a regular CW complex homeomorphic to a ball. That is, the closure of every cell is homeomorphic to a closed ball.

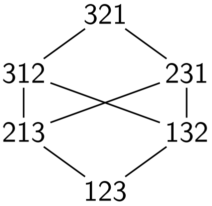
- e.g.  non-regular CW complex  regular CW complex  regular CW complex
- Williams (2007): The face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is *thin* and *shellable*. Thus it is the face poset of *some* regular CW complex homeomorphic to a ball.
- Postnikov, Speyer, Williams (2009): $\mathrm{Gr}_{k,n}^{\geq 0}$ is a CW complex (via *matching polytopes* of plabic graphs).
- Rietsch, Williams (2010): $\mathrm{Gr}_{k,n}^{\geq 0}$ is a regular CW complex up to homotopy (via discrete Morse theory).

Theorem (Galashin, Karp, Lam)

Postnikov's conjecture is true.

Motivation 1: combinatorics of regular CW complexes

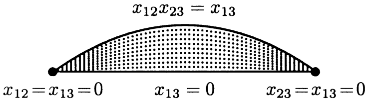
- Every convex polytope (decomposed into its open faces) is a regular CW complex. We can think of a regular CW complex as the 'next best thing' to a convex polytope.

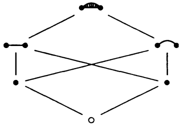


$\mathfrak{S}_3$ (strong order)

$$\mathrm{link}_{I_3}(U_3^{\geq 0}) = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : \begin{array}{l} x+z=1, \\ \text{all minors} \geq 0 \end{array} \right\}$$

$\leftsquigarrow$

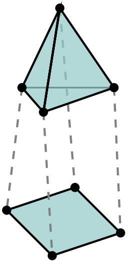




- Note that $\mathfrak{S}_n$ is not the face poset of a polytope. However, $\mathfrak{S}_n$ is shellable due to Edelman (1981), so it is the face poset of a regular CW complex homeomorphic to a ball, by work of Björner (1984).
- Bernstein, Björner: Is there such a regular CW complex 'in nature'?
- Fomin and Shapiro (2000) conjectured that $\mathrm{link}_{I_n}(U_n^{\geq 0})$ is such a regular CW complex. This was proved by Hersh (2014), in general Lie type.

Motivation 2: amplituhedra and Grassmann polytopes

- By definition, a polytope is the image of a simplex under an affine map:



simplex in $\mathbb{P}^{n-1}$ - - - - - - → $\mathrm{Gr}_{k,n}^{\geq 0}$

↓ linear map $\mathbb{R}^n \to \mathbb{R}^{m+1}$ ↓ linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$

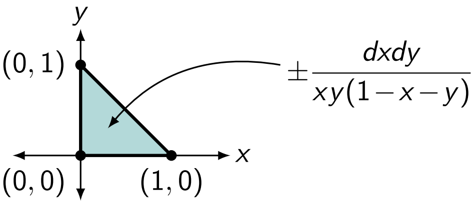
polytope in $\mathbb{P}^m$ - - - - - - → *Grassmann polytope* in $\mathrm{Gr}_{k,k+m}$

A *Grassmann polytope* is the image of a map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

- When the matrix Z has positive maximal minors, the Grassmann polytope is called an *amplituhedron*. Amplituhedra generalize cyclic polytopes ($k = 1$) and totally nonnegative Grassmannians ($k + m = n$). They were introduced by the physicists Arkani-Hamed and Trnka (2014), and inspired Lam (2015) to define Grassmann polytopes.

Motivation 2: amplituhedra and Grassmann polytopes

- Arkani-Hamed, Bai, Lam (2017): a *positive geometry* is a space equipped with a *canonical differential form*, which has logarithmic singularities at the boundaries of the space. Examples include convex polytopes:



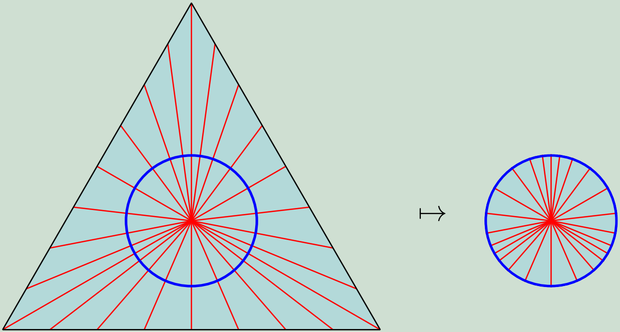
- The amplituhedron is conjecturally a positive geometry, whose canonical form for $m = 4$ is the tree-level scattering amplitude in planar $\mathcal{N} = 4$ SYM.
- Intuition from physics: the geometry determines the canonical form, and vice-versa. In order to understand amplituhedra (and more generally, Grassmann polytopes), we first need to understand $\mathrm{Gr}_{k,n}^{\geq 0}$.
- Other physically relevant positive geometries include *associahedra*, *cosmological polytopes*, *Cayley polytopes*, *halohedra*, *Stokes polytopes*, ...

Technique 1: contractive flows

Theorem

Every compact, convex subset of $\mathbb{R}^d$ is homeomorphic to a closed ball.

Proof



- This proof does not directly work for $\mathrm{Gr}_{k,n}^{\geq 0}$, since it is not *totally geodesic*.

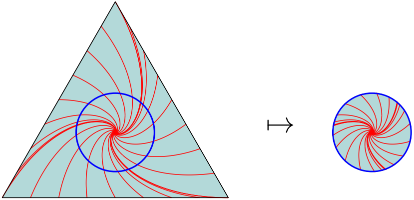
- Define the *(left) cyclic shift map* S on $\mathbb{R}^n$ by

$$S(x) := (x_2, x_3, \cdots, x_n, (-1)^{k-1}x_1) \quad \text{for } x = (x_1, \cdots, x_n) \in \mathbb{R}^n.$$

Then S gives the *cyclic action* on $\mathrm{Gr}_{k,n}$:

$$S \cdot \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | \\ v_2 & \cdots & v_n & (-1)^{k-1}v_1 \\ | & & | & | \end{bmatrix}.$$

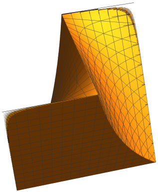
- We regard S as a vector field on $\mathrm{Gr}_{k,n}$, which sends $V \in \mathrm{Gr}_{k,n}$ along the trajectory $\exp(tS) \cdot V$ for $t \geq 0$. This contracts all of $\mathrm{Gr}_{k,n}^{\geq 0}$ onto the *attractor* $V_0 \in \mathrm{Gr}_{k,n}^{\geq 0}$, giving a homeomorphism onto a closed ball as before.
- e.g. $\mathrm{Gr}_{1,3}^{\geq 0}$

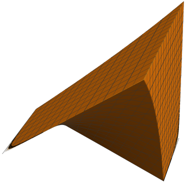


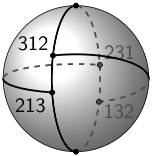
- A similar argument shows the following spaces are closed balls: *cyclically symmetric* amplituhedra, Lam's compactification of the space of electrical networks, and Lusztig's totally nonnegative partial flag varieties $(G/P)_{\geq 0}$.
- e.g. $\mathrm{Fl}_3^{\geq 0}$ consists of complete flags $\{0\} \subset W_1 \subset W_2 \subset \mathbb{R}^3$ such that W_1 and W_2 are totally nonnegative subspaces. This means that W_1 is spanned by a vector (x_1, x_2, x_3) and W_2 is orthogonal to a vector $(y_1, -y_2, y_3)$ with

$$x_1y_1 - x_2y_2 + x_3y_3 = 0, \quad x_1, x_2, x_3, y_1, y_2, y_3 \geq 0.$$

This space has 4 facets, given by setting one of x_1, y_1, x_3, y_3 to 0.







- Lusztig (1994), Rietsch (1999): $\mathrm{Fl}_n^{\geq 0}$ has a cell decomposition whose d-dimensional cells are indexed by intervals of length d in $(\mathfrak{S}_n, \leq_{\text{strong}})$.

Technique 2: links

- Unfortunately, there exist cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ with no *smooth* contractive flow.
- Brown (1962), Smale (1961), Freedman (1982), Perelman (2003):

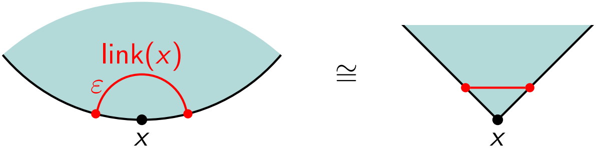
Theorem (consequence of generalized Poincaré conjecture)

Suppose that X is a compact topological manifold with boundary, whose interior X° is contractible and whose boundary ∂X is homeomorphic to a sphere. Then X is homeomorphic to a closed ball.

- We want to show that the closure X of a positroid cell in the cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a closed ball.
- Postnikov (2006): X° is homeomorphic to an open ball.
- By induction, we can assume that every cell closure in the boundary of X is homeomorphic to a closed ball, i.e. ∂X is a regular CW complex.
- Williams (2007): The face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is thin and shellable, so it is the face poset of a sphere. By Björner (1984), the homeomorphism type of a regular CW complex is determined by its face poset. Therefore by induction, ∂X is homeomorphic to a sphere.

Technique 2: links

- It remains to show that X is a topological manifold with boundary, i.e. X looks like a closed half-space in $\mathbb{R}^d$ near any point on its boundary.
- We adopt the framework of *links* from Fomin and Shapiro (2000), which they introduced to study the topology of $U_n^{\geq 0}$.



- We prove that:
  1. link(x) is homeomorphic to a closed ball;
  2. locally near x, the space X looks like the cone over link(x).

This implies that X is a topological manifold with boundary near x.
- We prove (1) by a similar induction. This does not reduce to a third induction, since 'links in links are links'.
- We introduce two key ideas: using *Snider's embedding*, and constructing a *dilation action* on the small spheres centered at x.

Snider's embedding

- The link framework of Fomin and Shapiro for $U_n^{\geq 0}$ involves factorization maps that do not have a direct analogue in $\mathrm{Gr}_{k,n}$. To get around this, we employ a construction of Snider (2011).
- We fix an index I, and embed the subset of $\mathrm{Gr}_{k,n}$ where $\Delta_I \neq 0$ into the *affine flag variety* $\widetilde{\mathrm{Fl}}_n$. We can think of $\widetilde{\mathrm{Fl}}_n$ as *n-periodic bi-infinite matrices* modulo left multiplication by invertible lower-triangular matrices.
- e.g. Let $I = \{1, 3\}$ with $k = 2, n = 4$. Then Snider's embedding is

$$\begin{bmatrix} 1 & a & 0 & b \\ 0 & c & 1 & d \end{bmatrix} \quad \mapsto \quad \begin{array}{cccccccccccc} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & \\ \cdots & 0 & a & 0 & b & 1 & 0 & \cdots & & & & \\ & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & & & \\ & & \cdots & 0 & d & 0 & c & 1 & 0 & \cdots & & \\ & & & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \\ & & & & \cdots & 0 & a & 0 & b & 1 & 0 & \cdots \\ & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array} .$$

- We can then apply the Fomin–Shapiro framework in $\widetilde{\mathrm{Fl}}_n$.
- We obtain the conic structure near x by translating x to a 'hidden' point in $\widetilde{\mathrm{Fl}}_n$ in the same cell as x, which does not come from a point in $\mathrm{Gr}_{k,n}$.

Open problems

Conjecture (Williams (2007))

The cell decomposition of $(G/P)_{\geq 0}$ *is a regular CW complex homeomorphic to a ball.*

- Show that Lam's compactification of the space of electrical networks forms a regular CW complex. Its face poset is the *uncrossing order on matchings*, which is Eulerian due to Lam (2015) and shellable due to Hersh and Kenyon (2018).
- Study the topology of amplituhedra and, more generally, Grassmann polytopes.

Thank you!