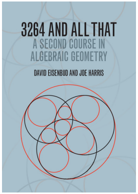
Wronskians, total positivity, and real Schubert calculus

Slides available at `snkarp.github.io`

Steven N. Karp (University of Notre Dame)
arXiv:2110.02301

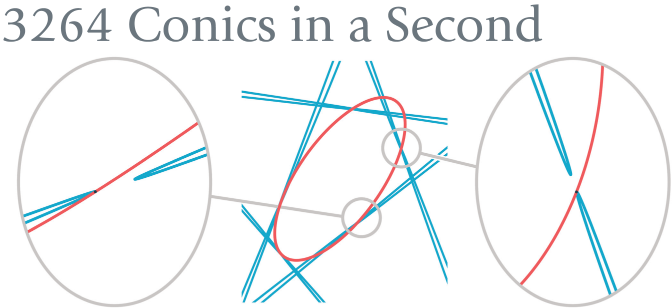
September 6th, 2022
University of Notre Dame

Steiner's conic problem (1848)



- How many conics are tangent to 5 given conics? ~~7776~~.
- de Jonquières (1859): 3264.
- Fulton (1996): "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems."

- Fulton (1986); Ronga, Tognoli, Vust (1997): All 3264 conics can be real.



- Breiding, Sturmfels, and Timme (2020) found 5 explicit such conics.

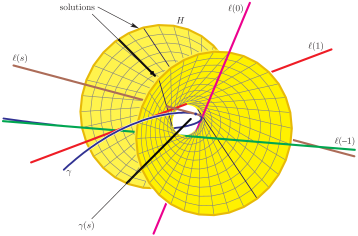
Shapiro–Shapiro conjecture (1995)

- Let $\mathrm{Gr}_{k,n}(\mathbb{C})$ be the *Grassmannian* of all k-dimensional subspaces of $\mathbb{C}^n$.
- Schubert (1886): Fix generic elements $W_1, \dots, W_{k(n-k)} \in \mathrm{Gr}_{k,n}(\mathbb{C})$.

Then there are $d_{k,n}$ elements $U \in \mathrm{Gr}_{n-k,n}(\mathbb{C})$ such that

$$U \cap W_i \neq \{0\} \text{ for all } i, \quad \text{where } d_{k,n} := \frac{1!2!\cdots(k-1)!}{(n-k)!(n-k+1)!\cdots(n-1)!}(k(n-k))!.$$

- B. and M. Shapiro conjectured that if each W_i is an osculating plane to the *rational normal curve* $\gamma(x) := (1, x, \dots, x^{n-1})$, then every U is real.
- e.g. $k=2$, $n=4$



F. Sottile, "Frontiers of reality in Schubert calculus"

- Bürgisser, Lerario (2020): a 'random' problem has $\approx\sqrt{d_{k,n}}$ real solutions.

Wronski map

- The *Wronskian* of k linearly independent functions $f_1, \ldots, f_k : \mathbb{C} \to \mathbb{C}$ is

$$\mathrm{Wr}(f_1, \ldots, f_k) := \det \begin{bmatrix} f_1 & \cdots & f_k \\ f_1' & \cdots & f_k' \\ \vdots & \ddots & \vdots \\ f_1^{(k-1)} & \cdots & f_k^{(k-1)} \end{bmatrix}.$$

- e.g. $\mathrm{Wr}(f, g) = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - f'g = f^2(\frac{g}{f})'$.
- Let $V := \mathrm{span}(f_1, \ldots, f_k)$. Then $\mathrm{Wr}(V)$ is well-defined up to a scalar. Its zeros are points in $\mathbb{C}$ where some nonzero $f \in V$ has a zero of order k.
- The monic linear differential operator $\mathcal{L}$ of order k with kernel V is

$$\mathcal{L}(g) = \frac{\mathrm{Wr}(f_1, \ldots, f_k, g)}{\mathrm{Wr}(f_1, \ldots, f_k)} = \frac{d^k g}{dx^k} + \cdots .$$

- We identify $\mathbb{C}^n$ with the space of polynomials of degree at most $n-1$:

$$\mathbb{C}^n \leftrightarrow \mathbb{C}[x]_{\leq n-1}, \quad (a_1, \ldots, a_n) \leftrightarrow a_1 + a_2 x + \cdots + a_n x^{n-1}.$$

We obtain the *Wronski map* $\mathrm{Wr} : \mathrm{Gr}_{k,n}(\mathbb{C}) \to \mathbb{P}(\mathbb{C}[x]_{\leq k(n-k)})$.

Wronskian formulation

Conjecture (Shapiro–Shapiro (1995))

Let $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$. If all complex zeros of $\mathrm{Wr}(V)$ are real, then V is real.

- e.g. If $\mathrm{Wr}(V) := (x+a)^2(x+b)^2$, the two solutions $V \in \mathrm{Gr}_{2,4}(\mathbb{C})$ are

  $$\mathrm{span}\big((x+a)(x+b), x(x+a)(x+b)\big) \text{ and } \mathrm{span}\big((x+a)^3, (x+b)^3\big).$$

- Sottile (1999) proved the conjecture asymptotically.
- Eremenko and Gabrielov (2002) proved the conjecture for $k = 2, n-2$.
- Mukhin, Tarasov, and Varchenko (2009) proved the conjecture via the *Bethe ansatz*. The proof was simplified by Purbhoo (2022).
- Purbhoo (2010) proved the Shapiro–Shapiro conjecture for the orthogonal Grassmannian. Analogues due to Sottile for the Lagrangian Grassmannian and the complete flag variety remain open.
- Levinson and Purbhoo (2021) proved the Shapiro–Shapiro conjecture topologically, and extended it to Wronskians with nonreal zeros.

Secant conjecture and disconjugacy conjecture

Conjecture (García-Puente, Hein, Hillar, Martín del Campo, Ruffo, Sottile, Teitler (2012))

Let $W_1, \ldots, W_{k(n-k)} \in \mathrm{Gr}_{k,n}(\mathbb{C})$*, where each* W_i *is spanned by* k *points on the rational normal curve* γ*, such that the points chosen for each* W_i *lie in* $k(n-k)$ *disjoint intervals of* $\mathbb{R}$*. Then all* $U \in \mathrm{Gr}_{n-k,n}(\mathbb{C})$ *satisfying*

$$U \cap W_i \neq \{0\} \text{ for all } i$$

are real.

- Eremenko (2015) showed that the secant conjecture is implied by:

Conjecture (Eremenko (2015))

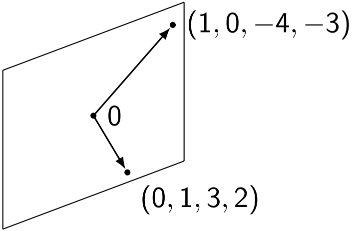
Let $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$*. If all zeros of* $\mathrm{Wr}(V)$ *are real, then every nonzero* $f \in V$ *has at most* $k-1$ *zeros in any interval of* $\mathbb{R}$ *on which* $\mathrm{Wr}(V)$ *is nonzero.*

- The case $k = 2$ of both conjectures was proved by Eremenko, Gabrielov, Shapiro, and Vainshtein (2006).

Total positivity

- Given $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$, take a $k \times n$ matrix whose rows span V. For k-element subsets I of $\{1, \dots, n\}$, let $\Delta_I(V)$ be the $k \times k$ minor located in columns I. The $\Delta_I(V)$ are well-defined up to a scalar, and give projective coordinates on $\mathrm{Gr}_{k,n}(\mathbb{C})$, called *Plücker coordinates*.
- e.g.

$V :=$



$$\in \mathrm{Gr}_{2,4}(\mathbb{C}) \quad \leftrightarrow \quad \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

$$\Delta_{12} = 1, \;\; \Delta_{13} = 3, \;\; \Delta_{14} = 2, \;\; \Delta_{23} = 4, \;\; \Delta_{24} = 3, \;\; \Delta_{34} = 1$$

- We say that $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$ is *totally nonnegative* if $\Delta_I(V) \geq 0$ for all I, and *totally positive* if $\Delta_I(V) > 0$ for all I.

Total positivity conjecture

Conjecture (Eremenko (2015))

Let $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$. If all zeros of $\mathrm{Wr}(V)$ are real, then every nonzero $f \in V$ has at most $k-1$ zeros in any interval of $\mathbb{R}$ on which $\mathrm{Wr}(V)$ is nonzero.

Conjecture (Mukhin, Tarasov (2017); Karp (2021))

Let $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$.
(i) If all zeros of $\mathrm{Wr}(V)$ lie in $[-\infty, 0]$, then V is totally nonnegative.
(ii) If all zeros of $\mathrm{Wr}(V)$ lie in $(-\infty, 0)$, then V is totally positive.

- e.g. Let $\mathrm{Wr}(V) := (x+a)^2(x+b)^2$. If $a, b > 0$, then the two solutions

$$\begin{bmatrix} ab & a+b & 1 & 0 \\ 0 & ab & a+b & 1 \end{bmatrix} \text{ and } \begin{bmatrix} a^3 & 3a^2 & 3a & 1 \\ b^3 & 3b^2 & 3b & 1 \end{bmatrix} \text{ are totally positive.}$$

Theorem (Karp (2021))

The two conjectures above are equivalent. They imply a totally positive version of the secant conjecture.

Complete flag variety

- The equivalence of the two conjectures follows from a new description of the totally positive part of the *complete flag variety* $\mathrm{Fl}_n(\mathbb{R})$.
- The elements of $\mathrm{Fl}_n(\mathbb{R})$ are tuples $(V_1, \dots, V_{n-1})$, where

$$V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{R}^n \quad \text{and } \dim(V_k) = k \text{ for all } 1 \le k \le n-1.$$

We say that $(V_1, \dots, V_{n-1})$ is *totally nonnegative* if all its Plücker coordinates are nonnegative, i.e., $V_k \in \mathrm{Gr}_{k,n}(\mathbb{R})$ is totally nonnegative for all $1 \le k \le n-1$. We similarly define *totally positive* complete flags.

Theorem (Karp (2021))

(i) The complete flag $(V_1, \dots, V_{n-1})$ *is totally nonnegative if and only if* $\mathrm{Wr}(V_k)$ *is nonzero on the interval* $(0, \infty)$*, for all* $1 \le k \le n-1$.
(ii) The complete flag $(V_1, \dots, V_{n-1})$ *totally positive if and only if* $\mathrm{Wr}(V_k)$ *is nonzero on the interval* $[0, \infty]$*, for all* $1 \le k \le n-1$.

- In the language of Chebyshev systems, the conclusions above say that $(V_1, \dots, V_{n-1})$ forms a *Markov system* (or *ECT-system*). Such systems also appear in the study of disconjugate linear differential equations.

Complete flag variety

- e.g. Let $n := 3$, and let $(V_1, V_2) \in \mathrm{Fl}_3(\mathbb{R})$ be represented by the matrix

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{with} \quad \begin{array}{l} \Delta_1 = 1,\ \Delta_2 = a,\ \Delta_3 = b, \\ \Delta_{12} = 1,\ \Delta_{13} = c,\ \Delta_{23} = ac - b. \end{array}$$

Hence (V_1, V_2) is totally positive if and only if $a, b, c, ac - b > 0$. Now,

$$\mathrm{Wr}(V_1) = \mathrm{Wr}(1 + ax + bx^2) = 1 + ax + bx^2,$$
$$\mathrm{Wr}(V_2) = \mathrm{Wr}(1 + ax + bx^2, x + cx^2) = 1 + 2cx + (ac - b)x^2.$$

The Theorem says that $a, b, c, ac - b > 0$ if and only if $\mathrm{Wr}(V_1)$ and $\mathrm{Wr}(V_2)$ are positive on $[0, \infty]$. The forward direction is immediate, and we can verify the reverse direction by a calculation.

- In general, the reverse direction follows using a topological argument.
- The Theorem also gives new total nonnegativity and total positivity tests for $\mathrm{Fl}_n(\mathbb{R})$ using the coefficients of the Wronskians. These lead to new total nonnegativity and total positivity tests for $\mathrm{GL}_n(\mathbb{R})$.