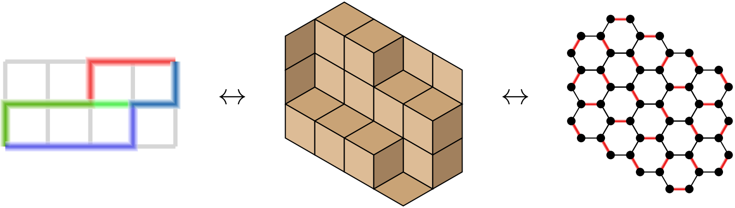
Combinatorics of the amplituhedron

Slides available at `www-personal.umich.edu/~snkarp`



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arXiv:1608.08288 (joint with Lauren Williams)
arXiv:1708.09525 (joint with Lauren Williams and Yan Zhang)
arXiv:1707.02010 (joint with Pavel Galashin and Thomas Lam)

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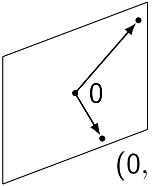
The Ohio State University

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

$V :=$ 

$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

$$\Delta_{12} = 1, \quad \Delta_{13} = 3, \quad \Delta_{14} = 2, \quad \Delta_{23} = 4, \quad \Delta_{24} = 3, \quad \Delta_{34} = 1$$

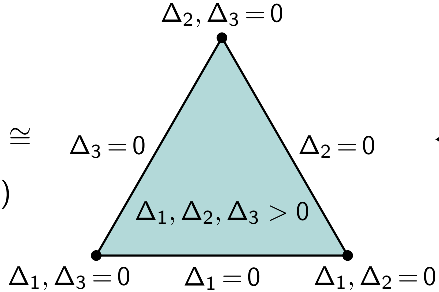
- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \cdots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call $V \in \mathrm{Gr}_{k,n}$ *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all such V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.
- When $k = 1$, the Grassmannian $\mathrm{Gr}_{1,n}$ specializes to projective space $\mathbb{P}^{n-1}$, the set of nonzero vectors $(x_1 : \cdots : x_n)$ modulo rescaling.

The 'faces' of $\mathrm{Gr}_{k,n}^{\geq 0}$

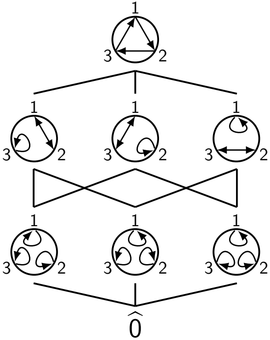
- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition due to Rietsch (1999) and Postnikov (2007). Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

$\mathrm{Gr}_{1,3}^{\geq 0} \cong$

$(= \mathbb{P}_{\geq 0}^{2})$



$\longleftrightarrow$



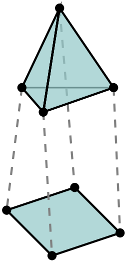
- $\mathrm{Gr}_{1,n}^{\geq 0}$ is an $(n-1)$-dimensional simplex in $\mathbb{P}^{n-1}$, since it equals

$$\{(x_1 : \cdots : x_n) \in \mathbb{P}^n \,:\, x_1, \cdots, x_n \geq 0,\;\; x_1 + \cdots + x_n = 1\}.$$

We can view $\mathrm{Gr}_{k,n}^{\geq 0}$ as a generalization of a simplex into the Grassmannian.

Amplituhedra and Grassmann polytopes

- By definition, a polytope is the image of a simplex under an affine map:



simplex in $\mathbb{P}^{n-1}$ - - - - → $\mathrm{Gr}_{k,n}^{\geq 0}$

linear map $\mathbb{R}^n \to \mathbb{R}^{m+1}$ (↓) | linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$ (↓)

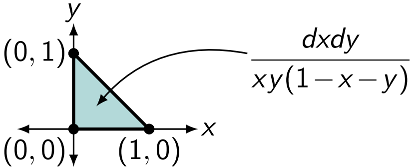
polytope in $\mathbb{P}^m$ - - - - → *Grassmann polytope* in $\mathrm{Gr}_{k,k+m}$

A *Grassmann polytope* is the image of a map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

- When the matrix Z has positive maximal minors, the Grassmann polytope is called an *amplituhedron*, denoted $\mathcal{A}_{n,k,m}(Z)$. Amplituhedra generalize cyclic polytopes ($k = 1$) and totally nonnegative Grassmannians ($k + m = n$). They were introduced by Arkani-Hamed and Trnka (2014), and inspired Lam (2015) to define Grassmann polytopes.

Positive geometries and canonical forms

- Arkani-Hamed, Bai, Lam (2017): a *positive geometry* is a space equipped with a canonical differential form, which has logarithmic singularities at the boundaries of the space. Examples include convex polytopes:



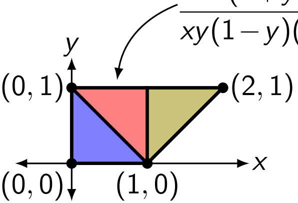
- $\mathrm{Gr}_{k,n}^{\geq 0}$ is a positive geometry. The canonical form of e.g. $\mathrm{Gr}_{2,4}^{\geq 0}$ is

$$\frac{dxdydzdw}{\Delta_{12}\Delta_{23}\Delta_{34}\Delta_{14}}, \text{ where } V = \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{bmatrix} \in \mathrm{Gr}_{2,4}\,.$$

- The amplituhedron is conjecturally a positive geometry, whose canonical form for $m = 4$ is the tree-level scattering amplitude in planar $\mathcal{N} = 4$ SYM.
- Other physically relevant positive geometries include *associahedra*, *cosmological polytopes*, *Cayley polytopes*, *halohedra*, *Stokes polytopes*, ...

Triangulations

- One way to find the canonical form of a positive geometry is by triangulating it into simpler pieces:



$$\frac{(1+y)dxdy}{xy(1-y)(1-x+y)} = \frac{dxdy}{xy(1-x-y)} + \frac{dxdy}{(1-x)(1-y)(x+y-1)} + \frac{dxdy}{(x-1)(1-y)(1-x+y)}$$

Conjecture (Arkani-Hamed, Trnka (2014))

The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is 'triangulated' by the images of certain $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, coming from the BCFW recursion.

Problem

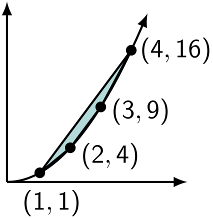
Find a 'triangulation-independent' description of the amplituhedron form.

Cyclic polytopes ($k = 1$)

- A *cyclic polytope* is a polytope (up to combinatorial equivalence) whose vertices lie on the *moment curve*

$$f(t) := (t, t^2, \ldots, t^m) \text{ in } \mathbb{R}^m \quad (t > 0).$$

- e.g. $m = 2$



$$\longleftrightarrow \qquad Z = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix}$$

We can identify this polytope with the amplituhedron $\mathcal{A}_{4,1,2}(Z) \subseteq \mathrm{Gr}_{1,3}$.

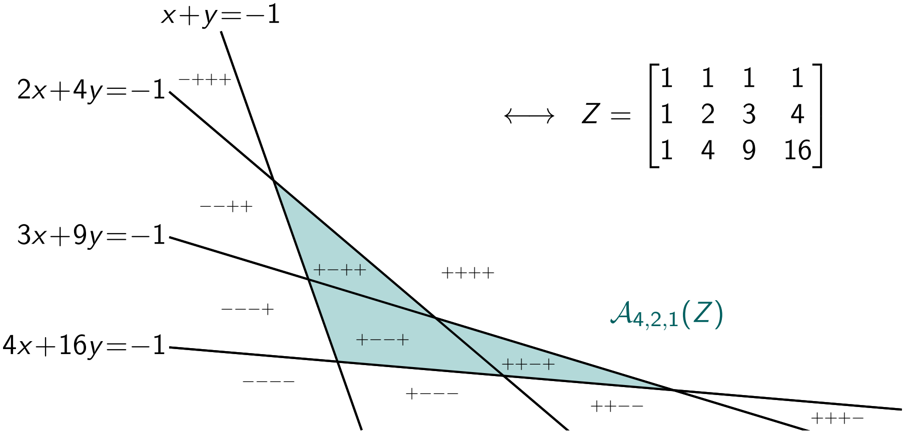
- Sturmfels (1988): every amplituhedron $\mathcal{A}_{n,1,m}(Z)$ is a cyclic polytope. The columns of the $(m+1) \times n$ matrix Z give the vertices in $\mathbb{P}^m$.
- According to Arkani-Hamed and Trnka's conjecture, the BCFW recursion triangulates any 4-dimensional cyclic polytope. Indeed, such triangulations were studied by Rambau (1997).

Cyclic hyperplane arrangements ($m = 1$)

- A *cyclic hyperplane arrangement* consists of n hyperplanes of the form

$$tx_1 + t^2 x_2 + \cdots + t^k x_k + 1 = 0 \text{ in } \mathbb{R}^k \quad (t > 0).$$

- e.g. $k = 2$, $n = 4$



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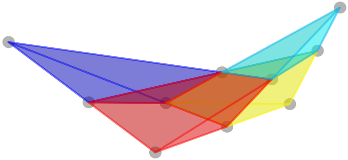
Cyclic hyperplane arrangements ($m = 1$)

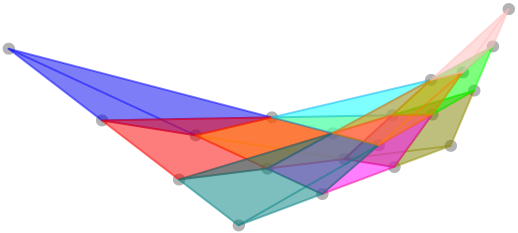
Theorem (Karp, Williams)

(i) $\mathcal{A}_{n,k,1}(Z)$ *is isomorphic to the complex of bounded faces of a cyclic hyperplane arrangement of* n *hyperplanes in* $\mathbb{R}^k$.
(ii) $\mathcal{A}_{n,k,1}(Z)$ *is isomorphic to a subcomplex of cells of* $\mathrm{Gr}_{k,n}^{\geq 0}$.
(iii) $\mathcal{A}_{n,k,1}(Z)$ *is homeomorphic to a closed ball of dimension* k.

- e.g.



$\mathcal{A}_{5,3,1}(Z)$



$\mathcal{A}_{6,3,1}(Z)$

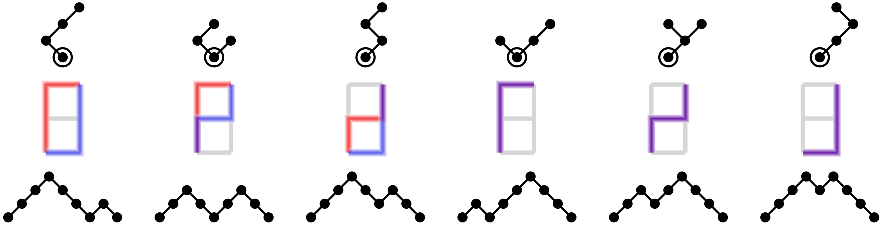
- Part (iii) follows from a general result of Dong (2008) about the complex of bounded faces of *any* hyperplane arrangement in general position.

BCFW triangulation for $m = 4$

Conjecture (Arkani-Hamed, Trnka (2014))

The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is 'triangulated' by the images of certain $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, coming from the BCFW recursion.

- The number of top-dimensional cells in a BCFW triangulation is the *Narayana number* $N_{n-3,k+1} := \frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$.
- e.g. For $n = 7$, $k = 2$, we have $N_{7-3,2+1} = 6$:



- $k = 1$, m even: every triangulation of $\mathcal{A}_{n,1,m}(Z)$ has $\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$ top cells.
- $m = 2$: there is a nice triangulation of $\mathcal{A}_{n,k,2}(Z)$ with $\binom{n-2}{k}$ top cells.

Number of cells in a triangulation

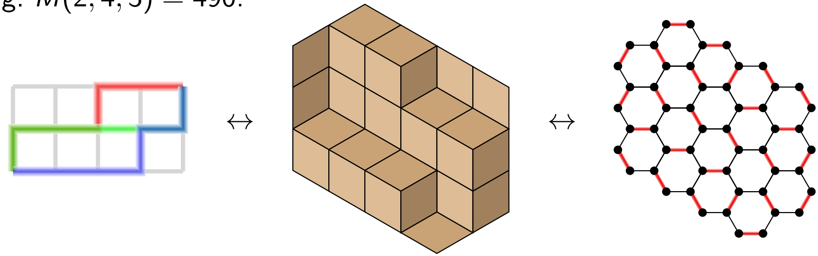
- Define the *MacMahon number*

$$M(a,b,c) := \prod_{p=1}^{a}\prod_{q=1}^{b}\prod_{r=1}^{c}\frac{p+q+r-1}{p+q+r-2}.$$

Conjecture (Karp, Williams, Zhang)

For m even, there exists a cell decomposition of $\mathcal{A}_{n,k,m}(Z)$ with $M(k, n-k-m, \frac{m}{2})$ top-dimensional cells.

- $M(a,b,c)$ is the number of *plane partitions* inside an $a \times b \times c$ box.
- e.g. $M(2,4,3) = 490$:



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Problem

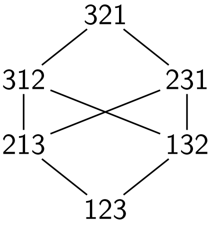
Interpret properties of plane partitions in terms of amplituhedra.

- The $k \leftrightarrow n-k-m$ symmetry comes (for $m=4$) from *parity* of the scattering amplitude. Galashin and Lam (2018) showed that the *stacked twist map* interchanges triangulations of $\mathcal{A}_{n,k,m}(Z)$ and $\mathcal{A}_{n,n-k-m,m}(Z')$.

Problem

Explain the conjectural symmetry for amplituhedra between k and $\frac{m}{2}$.

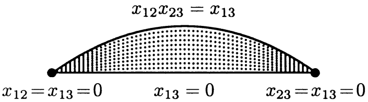
Topology of totally positive spaces

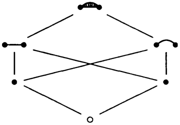


$\mathfrak{S}_3$ (Bruhat order)

$\leftrightsquigarrow$

$$\mathrm{link}_{I_3}(U_3^{\geq 0}) = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : \begin{array}{l} x + z = 1, \\ \text{all minors} \geq 0 \end{array} \right\}$$





- Edelman (1981) showed that $\mathfrak{S}_n$ is *shellable*. By work of Björner (1984), this implies that $\mathfrak{S}_n$ is the face poset of some *regular CW complex* homeomorphic to a ball, the "next best thing" to a convex polytope.
- Bernstein, Björner: Is there such a regular CW complex 'in nature'?
- Lusztig (1994): $U_n^{\geq 0}$ has a cell decomposition whose face poset is $\mathfrak{S}_n$.
- Fomin and Shapiro (2000) conjectured that $\mathrm{link}_{I_n}(U_n^{\geq 0})$ is the desired regular CW complex. This was proved by Hersh (2014), in all Lie types.

The topology of $\mathrm{Gr}_{k,n}^{\geq 0}$ and Grassmann polytopes

Theorem (Galashin, Karp, Lam)

The cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ is a regular CW complex homeomorphic to a ball. Thus the closure of every cell is homeomorphic to a closed ball.

- The faces of an amplituhedron (or Grassmann polytope) are images under Z of the closed cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. Understanding the topology of these cells is an important step in developing a theory of Grassmann polytopes.

Problem

Show that any amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is homeomorphic to a closed ball.

Problem

Show that $\mathcal{A}_{n,k,m}(Z)$ is a regular CW complex.

Problem

Define a decomposition of $\mathcal{A}_{n,k,m}$, and show that its face poset is shellable.