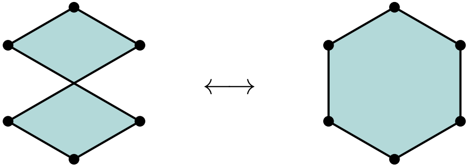
Gradient flows on totally nonnegative flag varieties

Slides available at `snkarp.github.io`



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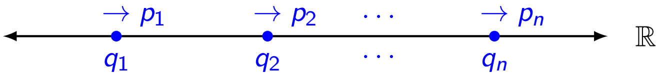
Toda lattice

- The *Toda lattice* (1967) is a Hamiltonian system with

$$H(\mathbf{q},\mathbf{p}) := \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} \qquad \left(\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i}\right).$$



- Flaschka (1974) expressed the Toda flow in *Lax form*: $\dot{L} = [L, \pi_{\text{skew}}(L)]$, where L is an $n \times n$ symmetric tridiagonal matrix with positive subdiagonal.

$$L = \begin{bmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \\ 0 & a_2 & b_3 \end{bmatrix}, \quad \pi_{\text{skew}}(L) = \begin{bmatrix} 0 & -a_1 & 0 \\ a_1 & 0 & -a_2 \\ 0 & a_2 & 0 \end{bmatrix}, \quad a_i = \tfrac{1}{2} e^{\frac{q_i - q_{i+1}}{2}}, \quad b_i = -\tfrac{1}{2} p_i.$$

- The eigenvalues of L are distinct and invariant under the Toda flow. As $t \to \pm\infty$, L approaches a diagonal matrix with sorted diagonal entries.

Explicit solution of the Toda lattice flow

- The Toda flow evolves on the *isospectral manifold* $\mathcal{J}_{\boldsymbol{\lambda}}^{>0}$ of all tridiagonal L with fixed spectrum $\boldsymbol{\lambda} = (\lambda_1 > \cdots > \lambda_n)$ and all off-diagonal $a_i > 0$.

Theorem (Moser (1975))

The map which sends $L \in \mathcal{J}_{\boldsymbol{\lambda}}^{>0}$ to the vector $(u_1, \dots, u_n)$ of first entries of its normalized eigenvectors is a homeomorphism onto $S_{>0}^{n-1}$. The Toda lattice flow is a gradient flow on projective space $\mathbb{P}^{n-1}(\mathbb{R})$:

$$\dot{u}_i = \lambda_i u_i \qquad \text{for } 1 \le i \le n.$$

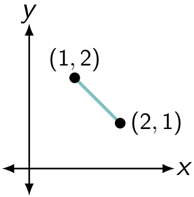
- e.g. $$L = \frac{1}{33}\begin{bmatrix} 50 & 28 & 0 \\ 28 & 81 & 8 \\ 0 & 8 & 67 \end{bmatrix} = \begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix} \in \mathcal{J}_{(3,2,1)}^{>0}$$

$$\mapsto (u_1, u_2, u_3) = \left(\tfrac{16}{33}, \tfrac{7}{33}, \tfrac{28}{33}\right) \in S_{>0}^2.$$

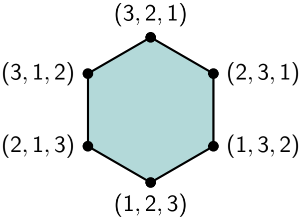
- Let $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$ denote the closure of $\mathcal{J}_{\boldsymbol{\lambda}}^{>0}$, where we allow $a_i = 0$.

Isospectral manifold $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$ and the permutohedron

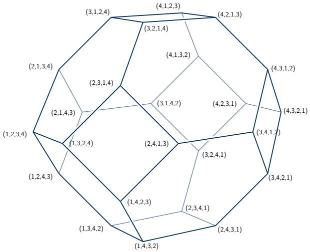
- Let $\mathrm{Perm}(\lambda_1, \dots, \lambda_n)$ be the polytope in $\mathbb{R}^n$ whose vertices are all $n!$ permutations of $(\lambda_1, \dots, \lambda_n)$, where $\lambda_1 > \cdots > \lambda_n$.



$\mathrm{Perm}(2, 1)$



$\mathrm{Perm}(3, 2, 1)$



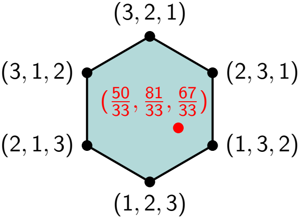
$\mathrm{Perm}(4, 3, 2, 1)$

Theorem (Tomei (1984))

The space $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$ is homeomorphic (as a stratified space) to $\mathrm{Perm}(\boldsymbol{\lambda})$.

Moment map and Schur–Horn theorem

- Let μ be the *moment map* sending a matrix to its diagonal.

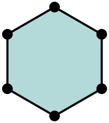
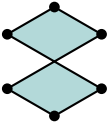


- e.g. $\mu\left(\frac{1}{33}\begin{bmatrix} 50 & 28 & 0 \\ 28 & 81 & 8 \\ 0 & 8 & 67 \end{bmatrix}\right) = (\frac{50}{33}, \frac{81}{33}, \frac{67}{33}) \in \mathbb{R}^3$.

Theorem (Schur (1923), Horn (1953))

The map μ sends the space of $n \times n$ symmetric matrices with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ onto $\mathrm{Perm}(\lambda_1, \dots, \lambda_n)$.

- However, $\mu : \mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0} \to \mathrm{Perm}(\boldsymbol{\lambda})$ is neither injective nor surjective.

- e.g. $\mathrm{Perm}(3, 2, 1) =$  $\mu(\mathcal{J}_{(3,2,1)}^{\geq 0}) =$ 

Twisted moment map

Theorem (Bloch, Flaschka, Ratiu (1990))

Let Λ be the diagonal matrix with diagonal $\boldsymbol{\lambda}$. The 'twisted moment map'

$$L = g\Lambda g^{-1} \mapsto \mu(g^{-1}\Lambda g) \qquad (g \in \mathrm{O}_n)$$

restricts to a homeomorphism $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0} \xrightarrow{\cong} \mathrm{Perm}(\boldsymbol{\lambda})$.

- e.g. $L = \begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix}$

$$\mapsto \mu\left(\begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \right) = \left(\frac{795}{363}, \frac{401}{363}, \frac{982}{363}\right).$$

- The proof defines a map $L = g\Lambda g^{-1} \mapsto g^{-1}\Lambda g$ on $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$, where $g \in \mathrm{O}_n$ depends smoothly on L. We use total positivity to (re-)construct this map.

Totally nonnegative flag variety

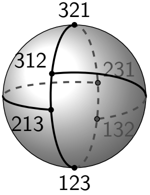
- The *complete flag variety* $\mathrm{Fl}_n(\mathbb{C})$ consists of all $V = (V_1, \dots, V_{n-1})$ with

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \quad \text{and } \dim(V_k) = k \text{ for all } k.$$

- We say $g \in \mathrm{GL}_n(\mathbb{C})$ *represents* V if each V_k is spanned by the first k columns of g. We call V *totally positive* (denoted $V \in \mathrm{Fl}_n^{>0}$) if we can find a g whose left-justified minors are all positive. We similarly define $\mathrm{Fl}_n^{\geq 0}$.

- e.g. $\begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{4} & 1 & 0 \\ \frac{7}{16} & \frac{17}{4} & 1 \end{bmatrix} \in \mathrm{Fl}_3^{>0}.$

- Lusztig (1994), Rietsch (1999): $\mathrm{Fl}_n^{\geq 0}$ has a cell decomposition.



$\mathrm{Fl}_3^{\geq 0}$

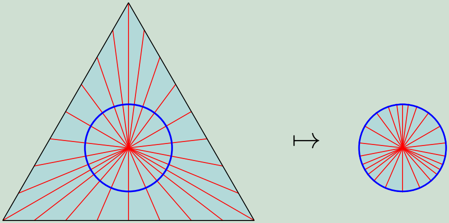
Contractive flow and topology of $\mathrm{Fl}_n^{\geq 0}$

Theorem (Galashin, Karp, Lam (2019))

The space $\mathrm{Fl}_n^{\geq 0}$ is homeomorphic to a closed Euclidean ball.

Proof

Let M be the $n \times n$ tridiagonal matrix $\begin{bmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$. Then $V \mapsto \exp(tM)V$ for $t \in [0, \infty]$ contracts $\mathrm{Fl}_n^{\geq 0}$ onto a unique attractor in the interior.



Totally nonnegative adjoint orbit

- Let U_n be the group of $n \times n$ unitary matrices and $\mathfrak{u}_n$ its Lie algebra of $n \times n$ skew-Hermitian matrices. For $\lambda_1 > \cdots > \lambda_n$, consider the adjoint orbit

$$\mathcal{O}_{\boldsymbol{\lambda}} := \{g(\mathrm{i}\Lambda)g^{-1} : g \in \mathrm{U}_n\} \subseteq \mathfrak{u}_n, \quad \text{where } \Lambda := \mathrm{Diag}(\lambda_1, \dots, \lambda_n).$$

We have the isomorphism

$$\mathcal{O}_{\boldsymbol{\lambda}} \xrightarrow{\cong} \mathrm{Fl}_n(\mathbb{C}), \quad g(\mathrm{i}\Lambda)g^{-1} \mapsto g,$$

sending a matrix to its flag of eigenvectors ordered by descending eigenvalue.

- We define $\mathcal{O}_{\boldsymbol{\lambda}}^{>0}$ and $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ to be the preimages of $\mathrm{Fl}_n^{>0}$ and $\mathrm{Fl}_n^{\geq 0}$.
- e.g. $\begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3\mathrm{i} & 0 & 0 \\ 0 & 2\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix} = \frac{\mathrm{i}}{33} \begin{bmatrix} 50 & 28 & 0 \\ 28 & 81 & 8 \\ 0 & 8 & 67 \end{bmatrix} \in \mathcal{O}_{(3,2,1)}^{>0}$.

Proposition (Bloch, Karp (2023))

The tridiagonal subset of $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ is precisely $\mathrm{i}\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$ (i.e. where all off-diagonal entries lie on the nonnegative imaginary axis).

Gradient flows on adjoint orbits

- We consider the gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ of the function $L \mapsto 2n\,\mathrm{tr}(LN)$, where $N \in \mathfrak{u}_n$. We work in the *Kähler*, *normal*, and *induced* metrics.
- Kähler metric: the gradient flow is $V(t) = \exp(t\mathrm{i}N)V$ on $\mathrm{Fl}_n(\mathbb{C}) \cong \mathcal{O}_{\boldsymbol{\lambda}}$.
- Normal metric: the gradient flow is $\dot{L} = [L, [L, N]]$ on $\mathcal{O}_{\boldsymbol{\lambda}}$.
- Induced metric: the gradient flow is $\dot{L} = [L, \mathrm{ad}_L^{-1}(N)]$ on $\mathcal{O}_{\boldsymbol{\lambda}}$.
- We say the flow *strictly preserves positivity* if trajectories starting in $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ lie in $\mathcal{O}_{\boldsymbol{\lambda}}^{>0}$ for all positive time. If so, we obtain a contractive flow.

Theorem (Bloch, Karp (2023))

(i) Kähler metric: the gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ with respect to N strictly preserves positivity if and only if $\mathrm{i}N \in \mathcal{J}_{\boldsymbol{\mu}}^{>0}$ for some $\boldsymbol{\mu}$.

(ii) Normal metric: no gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ strictly preserves positivity.

(iii) Induced metric: if $n = 3$ and $\frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_3} \notin [\frac{1}{2+2\sqrt{2}}, 2 + 2\sqrt{2}]$, then no gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ in the induced metric strictly preserves positivity.

Twist map and Toda flows

- Every element of $\mathrm{Fl}_n^{\geq 0}$ is represented by a unique $g \in \mathrm{U}_n$ whose left-justified minors are all nonnegative. Let $\vartheta(g) := \left((-1)^{i+j}(g^{-1})_{i,j}\right)_{1 \leq i,j \leq n}$.
- e.g. $\vartheta\left(\frac{1}{33}\begin{bmatrix} 16 & -7 & 28 \\ 28 & -4 & -17 \\ 7 & 32 & 4 \end{bmatrix}\right) = \frac{1}{33}\begin{bmatrix} 16 & -28 & 7 \\ 7 & -4 & -32 \\ 28 & 17 & 4 \end{bmatrix}$.

Theorem (Bloch, Karp (2023))

The twist map ϑ is an involutive diffeomorphism $\mathrm{Fl}_n^{\geq 0} \xrightarrow{\cong} \mathrm{Fl}_n^{\geq 0}$.

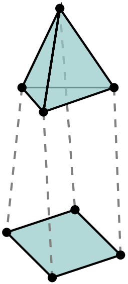
- The map ϑ induces a map on $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$. Restricting to $\mathrm{i}\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$, we recover the map of Bloch, Flaschka, and Ratiu on $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$ (i.e. $L = g\Lambda g^{-1} \mapsto g^{-1}\Lambda g$).

Theorem (Bloch, Karp (2023))

The full symmetric Toda flow $\dot{L} = [L, \pi_{\mathfrak{u}_n}(-\mathrm{i}L)]$ on $\mathcal{O}_{\boldsymbol{\lambda}}$ weakly preserves positivity in both time directions. It is the twisted gradient flow with respect to $N = -\mathrm{i}\Lambda$ in the Kähler metric.

Grassmann polytopes and amplituhedra

- The *Grassmannian* $\mathrm{Gr}_{k,n}(\mathbb{C})$ is the set of all k-dimensional subspaces of $\mathbb{C}^n$. Its totally nonnegative part $\mathrm{Gr}_{k,n}^{\geq 0}$ is the projection of $\mathrm{Fl}_n^{\geq 0}$. If $k = 1$ then $\mathrm{Gr}_{1,n}^{\geq 0}$ is a simplex in $\mathbb{P}^{n-1}$.
- A *Grassmann polytope* is the Grassmannian analogue of a polytope.



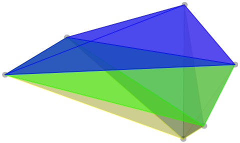
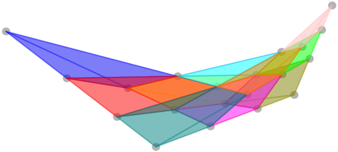
simplex in $\mathbb{P}^{n-1}$ - - - - - - - → $\mathrm{Gr}_{k,n}^{\geq 0}$

↓ linear map $\mathbb{R}^n \to \mathbb{R}^{m+1}$ ↓ linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$

polytope in $\mathbb{P}^m$ - - - - - - - → *Grassmann polytope* in $\mathrm{Gr}_{k,k+m}(\mathbb{C})$

- When Z is *positive*, the Grassmann polytope is called an *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$. It is the Grassmannian analogue of a cyclic polytope.

Gradient flows on amplituhedra

$\mathcal{A}_{6,1,3}(Z)$   $\mathcal{A}_{6,3,1}(Z)$

- The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ encodes a complex differential form on $\mathrm{Gr}_{k,k+m}(\mathbb{C})$. When $m = 4$, this form is (conjecturally) the tree-level scattering amplitude in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory.
- Intuition from physics: the combinatorics and geometry of $\mathcal{A}_{n,k,m}(Z)$ (e.g. triangulations, dualities) encode properties of the differential form.

Theorem (Bloch, Karp (2023))

Every 'twisted Vandermonde' amplituhedron $\mathcal{A}_{n,k,m}(Z)$ admits a contractive gradient flow, and is homeomorphic to a closed Euclidean ball. This includes all amplituhedra with $n - k - m \leq 2$.

- It is expected that every amplituhedron is homeomorphic to a closed ball.

Future directions

- Find contractive flows on the cell closures of $\mathrm{Fl}_n^{\geq 0}$.
- Find contractive flows on all amplituhedra.
- Generalize the connection to Toda flows to the periodic Toda lattice.
- Study Toda flows projected onto permutohedra.

Thank you!