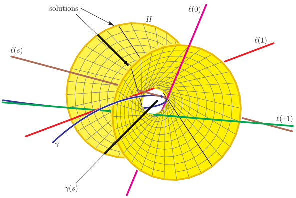
Positivity in real Schubert calculus

Slides available at `snkarp.github.io`



F. Sottile, "Frontiers of reality in Schubert calculus"

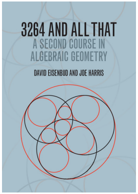
M. Griffon, CC BY 3.0 Deed

Steven N. Karp (University of Notre Dame)
joint work with Kevin Purbhoo

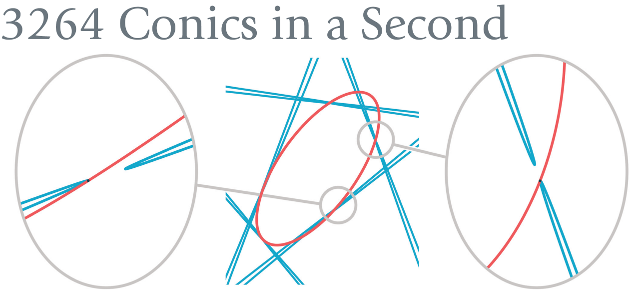
`arXiv:2309.04645`

March 27, 2024
Richard P. Stanley seminar in combinatorics

Steiner's conic problem (1848)



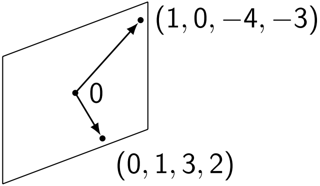
- How many conics are tangent to 5 given conics? ~~7776~~.
- de Jonquières (1859): 3264.
- Fulton (1984): "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems."
- Fulton (1986); Ronga, Tognoli, Vust (1997): All 3264 conics can be real.



- Breiding, Sturmfels, and Timme (2020) found 5 explicit such conics.

The Grassmannian $\mathrm{Gr}_{d,m}(\mathbb{C})$

- The *Grassmannian* $\mathrm{Gr}_{d,m}(\mathbb{C})$ is the set of d-dimensional subspaces of $\mathbb{C}^m$.

$V :=$ 

$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 6 & 4 \end{bmatrix} \in \mathrm{Gr}_{2,4}(\mathbb{C})$$

$$\Delta_{1,2} = 1, \quad \Delta_{1,3} = 3, \quad \Delta_{1,4} = 2, \quad \Delta_{2,3} = 4, \quad \Delta_{2,4} = 3, \quad \Delta_{3,4} = 1$$

$$\text{Plücker relation: } \Delta_{1,3}\Delta_{2,4} = \Delta_{1,2}\Delta_{3,4} + \Delta_{1,4}\Delta_{2,3}$$

- $\mathrm{Gr}_{d,m}(\mathbb{C}) \cong \mathrm{GL}_d(\mathbb{C}) \backslash \{d \times m \text{ matrices of rank } d\}$.

- Given $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ as a $d \times m$ matrix, for d-subsets I of $\{1, \dots, m\}$ let $\Delta_I(V)$ be the $d \times d$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well-defined up to a common scalar.

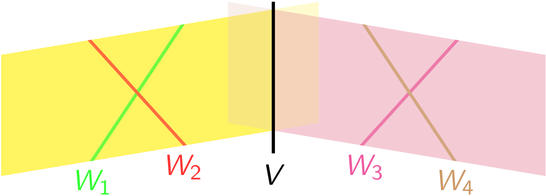
- $\mathrm{Gr}_{d,m}(\mathbb{C})$ is a projective variety of dimension $d(m-d)$.

Schubert calculus (1886)

- Divisor Schubert problem: given $W_1, \dots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{C})$, find all

$$V \in \mathrm{Gr}_{d,m}(\mathbb{C}) \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- e.g. $d = 2$, $m = 4$ (projectivized). Given 4 lines $W_i \subseteq \mathbb{P}^3$, find all lines $V \subseteq \mathbb{P}^3$ intersecting all 4. Generically, there are 2 solutions.



1	2
3	4

1	3
2	4

We can see the 2 solutions explicitly when two pairs of the lines intersect.

- If the W_i's are generic, the number of solutions V is

$$\#_{d,m} := \frac{1!2!\cdots(d-1)!}{(m-d)!(m-d+1)!\cdots(m-1)!}(d(m-d))!,$$

the number of *standard Young tableaux* of rectangular shape $d \times (m-d)$.

Shapiro–Shapiro conjecture (1993)

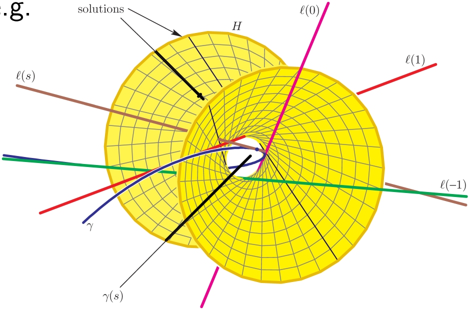
- Are there Schubert problems with all solutions real?

Shapiro–Shapiro conjecture (1993)

Let $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ osculate the moment curve $\gamma(t) := (1, t, \ldots, t^{m-1})$ at real points. Then there are $\#_{d,m}$ real solutions

$$V \in \mathrm{Gr}_{d,m}(\mathbb{R}) \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- e.g.



F. Sottile, "Frontiers of reality in Schubert calculus"

- The Schubert problem above arises in the study of linear series in algebraic geometry, differential equations, and pole placement problems in control theory.
- Bürgisser, Lerario (2020): a 'random' Schubert problem has $\approx\sqrt{\#_{d,m}}$ real solutions.

Shapiro–Shapiro conjecture (1993)

- Sottile (1999) tested the conjecture and proved it asymptotically.
- Eremenko and Gabrielov (2002) proved the conjecture for $d = 2, m - 2$.
- Mukhin, Tarasov, and Varchenko (2009) proved the full conjecture via the *Bethe ansatz*. The proof was simplified by Purbhoo (2022).
- Levinson and Purbhoo (2021) gave a topological proof of the conjecture.

Secant conjecture, divisor form (Sottile (2003))

Let $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ be secant to $\gamma(t) := (1, t, \ldots, t^{m-1})$ along non-overlapping intervals. Then there are $\#_{d,m}$ real solutions

$$V \in \mathrm{Gr}_{d,m}(\mathbb{R}) \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- The Shapiro–Shapiro conjecture is a limiting case of this conjecture.

Theorem (Karp, Purbhoo (2023))

The divisor form of the secant conjecture is true.

Total positivity

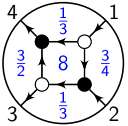
- Total positivity originated in the 1910's from orthogonal polynomials. Gantmakher and Krein (1937) showed that *totally positive matrices* (whose minors are all positive) have positive eigenvalues.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 10.6031\cdots \\ \lambda_2 &= 1.2454\cdots \\ \lambda_3 &= 0.1514\cdots \end{aligned}$$

- Lusztig (1994) introduced total positivity for algebraic groups and flag varieties, motivated by quantum groups. An element $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ is *totally nonnegative* if its Plücker coordinates are all nonnegative.

$$V := \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

$$\Delta_{1,2} = 1,\ \Delta_{1,3} = 3,\ \Delta_{1,4} = 2,\ \Delta_{2,3} = 4,\ \Delta_{2,4} = 3,\ \Delta_{3,4} = 1$$



- Postnikov (2006) parametrized $\mathrm{Gr}_{d,m}^{\geq 0}$ using *plabic graphs*.

- $\mathrm{Gr}_{d,m}^{\geq 0}$ appears in the study of cluster algebras, electrical networks, the KP hierarchy, scattering amplitudes, curve singularities, the Ising model, ...

Wronski map

- The *Wronskian* of d functions $f_1, \ldots, f_d : \mathbb{C} \to \mathbb{C}$ is

$$\mathrm{Wr}(f_1, \ldots, f_d) := \det \begin{bmatrix} f_1 & \cdots & f_d \\ f_1' & \cdots & f_d' \\ \vdots & \ddots & \vdots \\ f_1^{(d-1)} & \cdots & f_d^{(d-1)} \end{bmatrix}.$$

- e.g. $\mathrm{Wr}(f, g) = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - f'g = f^2(\frac{g}{f})'$.

- $\mathrm{Wr}(f_1, \ldots, f_d) \not\equiv 0$ if and only if $f_1, \ldots, f_d$ are linearly independent. Then $\mathrm{Wr}(V)$ is well-defined up to a scalar, where $V := \langle f_1, \ldots, f_d \rangle$.

- The monic linear differential operator $\mathcal{L}$ of order d with kernel V is

$$\mathcal{L}(g) = \frac{\mathrm{Wr}(f_1, \ldots, f_d, g)}{\mathrm{Wr}(f_1, \ldots, f_d)} = g^{(d)} + \cdots .$$

- We identify $\mathbb{C}^m$ with the space of polynomials of degree at most $m - 1$:

$$\mathbb{C}^m \leftrightarrow \mathbb{C}_{m-1}[u], \quad (a_1, \ldots, a_m) \leftrightarrow a_1 + a_2 u + a_3 \frac{u^2}{2} + \cdots + a_m \frac{u^{m-1}}{(m-1)!}.$$

We obtain the *Wronski map* $\mathrm{Wr} : \mathrm{Gr}_{d,m}(\mathbb{C}) \to \mathbb{P}(\mathbb{C}_{d(m-d)}[u])$.

Shapiro–Shapiro conjecture and positivity conjecture

Shapiro–Shapiro conjecture (Mukhin, Tarasov, and Varchenko (2009))

Let $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$. If all complex zeros of $\mathrm{Wr}(V)$ are real, then V is real.

- e.g. If $\mathrm{Wr}(V) := (u+z_1)^2(u+z_2)^2$, the two solutions $V \in \mathrm{Gr}_{2,4}(\mathbb{C})$ are

$$\big\langle (u+z_1)(u+z_2), u(u+z_1)(u+z_2) \big\rangle \quad \text{and} \quad \big\langle (u+z_1)^3, (u+z_2)^3 \big\rangle.$$

$$= \begin{bmatrix} z_1z_2 & z_1+z_2 & 2 & 0 \\ 0 & z_1z_2 & 2(z_1+z_2) & 6 \end{bmatrix} \qquad = \begin{bmatrix} z_1^3 & 3z_1^2 & 6z_1 & 6 \\ z_2^3 & 3z_2^2 & 6z_2 & 6 \end{bmatrix}$$

Positivity conjecture (Mukhin, Tarasov (2017); Karp (2021))

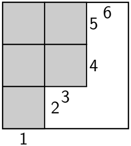
Let $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$. If all zeros of $\mathrm{Wr}(V)$ are nonpositive, then $V \in \mathrm{Gr}_{d,m}^{\geq 0}$.

- Karp (2023): The positivity conjecture is equivalent a conjecture of Eremenko (2015), which implies the divisor form of the secant conjecture.
- Karp, Purbhoo (2023): The positivity conjecture is true.

Universal Plücker coordinates

- We want to find all V with $\mathrm{Wr}(V) = (u + z_1) \cdots (u + z_n)$. It suffices to work in $\mathrm{Gr}_{n,2n}(\mathbb{C})$. We construct *universal* Plücker coordinates $\beta^\lambda \in \mathbb{C}[\mathfrak{S}_n]$ indexed by *partitions* λ.
- Partitions inside the $n \times n$ square index n-element subsets of $\{1, \ldots, 2n\}$.

$$\lambda = (2, 2, 1)$$
$$|\lambda| = 5$$
$$n = 3$$



$$\leftrightarrow \quad I_\lambda = \{2, 4, 5\}$$

Theorem (Karp, Purbhoo (2023))

(i) The β^λ's pairwise commute and satisfy the Plücker relations.

(ii) There is a bijection between the eigenspaces of the β^λ's acting on $\mathbb{C}[\mathfrak{S}_n]$ and the elements $V \in \mathrm{Gr}_{n,2n}(\mathbb{C})$ with $\mathrm{Wr}(V) = (u + z_1) \cdots (u + z_n)$, sending the eigenvalue of β^λ to the Plücker coordinate $\Delta_{I_\lambda}(V)$.

(iii) If $z_1, \ldots, z_n \geq 0$, then the β^λ's are positive semidefinite.

Definition of the universal Plücker coordinates

$$\beta^{\lambda} := \sum_{\substack{X \subseteq \{1,\dots,n\},\\ |X| = |\lambda|}} \Big(\prod_{i \notin X} z_i \Big) \sum_{\pi \in \mathfrak{S}_X} \chi^{\lambda}(\pi)\pi \in \mathbb{C}[\mathfrak{S}_n]$$

- e.g. $n = 2$. Write $\mathfrak{S}_2 = \{e, \sigma\}$, where e is the identity. We have

$$\beta^{\varnothing} = z_1 z_2 e, \qquad \beta^{\square} = (z_1 + z_2)e, \qquad \beta^{\square\square} = e + \sigma, \qquad \beta^{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} = e - \sigma,$$

and $\beta^{\lambda} = 0$ if $|\lambda| > 2$. On the eigenspace $\langle e - \sigma \rangle$, the eigenvalues are

$$\beta^{\varnothing} \rightsquigarrow z_1 z_2, \qquad \beta^{\square} \rightsquigarrow z_1 + z_2, \qquad \beta^{\square\square} \rightsquigarrow 0, \qquad \beta^{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \rightsquigarrow 2,$$

which are the Plücker coordinates of

$$V = \begin{bmatrix} \frac{z_1+z_2}{2} & 1 & 0 & 0 \\ -z_1 z_2 & 0 & 2 & 0 \end{bmatrix} = \left\langle \frac{z_1 + z_2}{2} + u,\ -z_1 z_2 + u^2 \right\rangle \in \mathrm{Gr}_{2,4}(\mathbb{C}).$$

We can check that

$$\mathrm{Wr}(V) = \det \begin{bmatrix} \frac{z_1+z_2}{2} + u & -z_1 z_2 + u^2 \\ 1 & 2u \end{bmatrix} = (u + z_1)(u + z_2).$$

KP hierarchy

- The *KP equation*

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + 3\frac{\partial^2 u}{\partial y^2} = 0$$

models shallow water waves. It is the first equation in the *KP hierarchy*.

M. Griffon, CC BY 3.0 Deed

- The solutions to the KP hierarchy are symmetric functions $\tau(\mathbf{x})$ in the variables $\mathbf{x} = (x_1, x_2, \dots)$ satisfying the *Hirota bilinear identity*:

$$[t^{-1}]\exp\Big(\sum_{k\geq 1}\tfrac{t^k}{k}\big(p_k(\mathbf{x}) - p_k(\mathbf{y})\big)\Big)\exp\Big(\sum_{k\geq 1} -t^{-k}\big(\tfrac{\partial}{\partial p_k(\mathbf{x})} - \tfrac{\partial}{\partial p_k(\mathbf{y})}\big)\Big)\tau(\mathbf{x})\tau(\mathbf{y}) = 0.$$

- Sato (1981): $\tau(\mathbf{x})$ satisfies the bilinear identity if and only if its coefficients in the Schur basis $s_\lambda(\mathbf{x})$ satisfy the Plücker relations.
- A key to our proof is showing $\sum_\lambda \beta^\lambda s_\lambda(\mathbf{x})$ satisfies the bilinear identity.

Future directions

- Further explore the connection to the KP hierarchy.
- Find necessary and sufficient inequalities on the Plücker coordinates of V for all complex zeros of $\mathrm{Wr}(V)$ to be nonpositive, generalizing the Aissen–Schoenberg–Whitney theorem in the case $\dim(V) = 1$. (The positivity conjecture implies that the inequalities $\Delta_I(V) \geq 0$ are necessary.)
- Address generalizations and variations of the Shapiro–Shapiro conjecture: the discriminant conjecture, the general form of the secant conjecture, the monotone conjecture, the total reality conjecture for convex curves, ...

Thank you!