Positivity in real Schubert calculus

Slides available at `snkarp.github.io`

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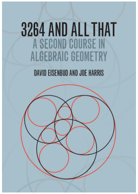
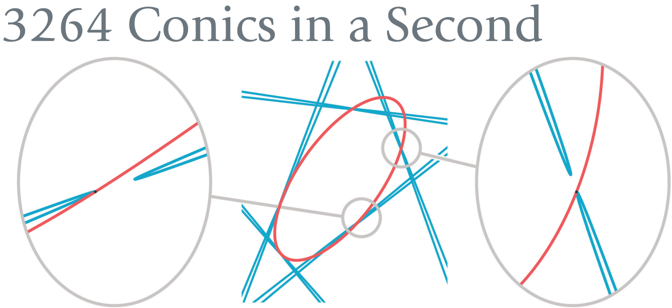
Steiner's conic problem (1848)



- How many conics are tangent to 5 given conics? ~~7776~~.
- de Jonquières (1859): 3264.
- Fulton (1984): "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems."
- Fulton (1986); Ronga, Tognoli, Vust (1997): All 3264 conics can be real.



- Breiding, Sturmfels, and Timme (2020) found 5 explicit such conics.

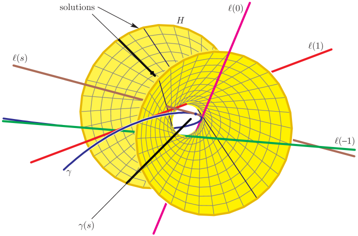
Shapiro–Shapiro conjecture (1995)

- Let $\mathrm{Gr}_{d,m}(\mathbb{C})$ be the *Grassmannian* of all d-dimensional subspaces of $\mathbb{C}^m$.
- Schubert (1886): Fix generic elements $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{C})$.

Then there are $f_{d,m}$ elements $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ such that

$$V \cap W_i \neq \{0\} \text{ for all } i, \quad \text{where } f_{d,m} := \frac{1!2!\cdots(d-1)!}{(m-d)!(m-d+1)!\cdots(m-1)!}(d(m-d))!.$$

- B. and M. Shapiro conjectured that if each W_i is an osculating plane to the *rational normal curve* $\gamma(x) := (1, x, \ldots, x^{m-1})$, then every V is real.
- e.g. $d = 2$, $m = 4$



F. Sottile, "Frontiers of reality in Schubert calculus"

- Bürgisser, Lerario (2020): a 'random' problem has $\approx\sqrt{f_{d,m}}$ real solutions.

Wronski map

- The *Wronskian* of d linearly independent functions $f_1, \ldots, f_d : \mathbb{C} \to \mathbb{C}$ is

$$\mathrm{Wr}(f_1, \ldots, f_d) := \det \begin{bmatrix} f_1 & \cdots & f_d \\ f_1' & \cdots & f_d' \\ \vdots & \ddots & \vdots \\ f_1^{(d-1)} & \cdots & f_d^{(d-1)} \end{bmatrix}.$$

- e.g. $\mathrm{Wr}(f, g) = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - f'g = f^2(\frac{g}{f})'$.
- Let $V := \langle f_1, \ldots, f_d \rangle$. Then $\mathrm{Wr}(V)$ is well-defined up to a scalar. Its zeros are points in $\mathbb{C}$ where some nonzero $f \in V$ has a zero of order d.
- The monic linear differential operator $\mathcal{L}$ of order d with kernel V is

$$\mathcal{L}(g) = \frac{\mathrm{Wr}(f_1, \ldots, f_d, g)}{\mathrm{Wr}(f_1, \ldots, f_d)} = g^{(d)} + \cdots .$$

- We identify $\mathbb{C}^m$ with the space of polynomials of degree at most $m-1$:

$$\mathbb{C}^m \leftrightarrow \mathbb{C}_{m-1}[u], \quad (a_1, \ldots, a_m) \leftrightarrow a_1 + a_2 u + a_3 \tfrac{u^2}{2} + \cdots + a_m \tfrac{u^{m-1}}{(m-1)!}.$$

We obtain the *Wronski map* $\mathrm{Wr} : \mathrm{Gr}_{d,m}(\mathbb{C}) \to \mathbb{P}(\mathbb{C}_{d(m-d)}[u])$.

Wronskian formulation

Shapiro–Shapiro conjecture (1993)

Let $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$. If all complex zeros of $\mathrm{Wr}(V)$ are real, then V is real.

- e.g. If $\mathrm{Wr}(V) := (u+z_1)^2(u+z_2)^2$, the two solutions $V \in \mathrm{Gr}_{2,4}(\mathbb{C})$ are

$$\big\langle (u+z_1)(u+z_2), u(u+z_1)(u+z_2) \big\rangle \quad \text{and} \quad \big\langle (u+z_1)^3, (u+z_2)^3 \big\rangle.$$

- Sottile (1999) tested the conjecture and proved it asymptotically.
- Eremenko and Gabrielov (2002) proved the conjecture for $d = 2, m - 2$. Thus rational functions with real critical points are real (up to equivalence).
- Mukhin, Tarasov, and Varchenko (2009) proved the conjecture via the *Bethe ansatz*. The proof was simplified by Purbhoo (2022).
- Purbhoo (2010) proved the analogue for the orthogonal Grassmannian. Analogues due to Sottile for other flag varieties remain open.
- Levinson and Purbhoo (2021) gave a topological proof of the conjecture.

Secant conjecture and disconjugacy conjecture

- A *secant plane* to the rational normal curve γ along the interval $I \subseteq \mathbb{R}$ is spanned by distinct points of γ on I.

Secant conjecture, divisor form (Sottile (2003))

Let $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ be secant planes to γ along disjoint real intervals. Then all solutions $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ to the divisor Schubert problem

$$V \cap W_i \neq \{0\} \text{ for all } i$$

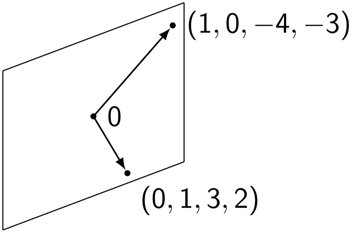
are real.

- The Shapiro–Shapiro conjecture is a limiting case of this conjecture.
- The general form of the secant conjecture involves arbitrary Schubert conditions. It was extensively tested extensively in a project led by Sottile.
- Eremenko (2015) posed a conjecture about *disconjugate* polynomials which implies the divisor form of the secant conjecture. The case $d = m-2$ was proved by Eremenko, Gabrielov, Shapiro, and Vainshtein (2006).

Total positivity

- Given $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$, take a $d \times m$ matrix whose rows span V. For d-element subsets I of $\{1, \ldots, m\}$, let $\Delta_I(V)$ be the $d \times d$ minor located in columns I. The $\Delta_I(V)$'s are well-defined up to a scalar, and give projective coordinates on $\mathrm{Gr}_{d,m}(\mathbb{C})$, called *Plücker coordinates*.
- e.g.

$V :=$



$$\in \mathrm{Gr}_{2,4}(\mathbb{C}) \quad \leftrightarrow \quad \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

$$\Delta_{12} = 1, \quad \Delta_{13} = 3, \quad \Delta_{14} = 2, \quad \Delta_{23} = 4, \quad \Delta_{24} = 3, \quad \Delta_{34} = 1$$

$$\text{Plücker relation}: \Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$$

- We say that $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ is *totally nonnegative* if $\Delta_I(V) \geq 0$ for all I.

Positivity conjecture

Positivity conjecture (Mukhin, Tarasov (2017); Karp (2021))

Let $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$. If all complex zeros of $\mathrm{Wr}(V)$ are nonpositive, then V is real and totally nonnegative.

- e.g. Let $\mathrm{Wr}(V) = (u+z_1)^2(u+z_2)^2$. If $z_1, z_2 \geq 0$, then the two solutions

$$\begin{bmatrix} z_1z_2 & z_1+z_2 & 2 & 0 \\ 0 & z_1z_2 & 2(z_1+z_2) & 6 \end{bmatrix} \text{ and } \begin{bmatrix} z_1^3 & 3z_1^2 & 6z_1 & 6 \\ z_2^3 & 3z_2^2 & 6z_2 & 6 \end{bmatrix} \text{ are totally nonnegative.}$$

Theorem (Karp (2021))

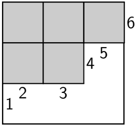
The positivity conjecture is equivalent to the disconjugacy conjecture. Hence it implies the divisor case of the secant conjecture.

Theorem (Karp, Purbhoo (2023))

These conjectures are true.

Universal Plücker coordinates

- We want to find all V with $\mathrm{Wr}(V) = (u + z_1) \cdots (u + z_n)$. It suffices to work in $\mathrm{Gr}_{n,2n}(\mathbb{C})$. We construct *universal* Plücker coordinates $\beta^\lambda \in \mathbb{C}[\mathfrak{S}_n]$.
- A *partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of positive integers. The *size* of λ is $|\lambda| := \lambda_1 + \cdots + \lambda_\ell$.
- e.g. $\lambda = (3, 2)$, $|\lambda| = 5$, $n = 3$



$\leftrightarrow \quad I_\lambda = \{1, 4, 6\}$

- Partitions inside the $n \times n$ square index n-element subsets of $\{1, \ldots, 2n\}$.

Theorem (Karp, Purbhoo (2023))

(i) The β^λ's pairwise commute and satisfy the Plücker relations.
(ii) There is a bijection between the eigenspaces of the β^λ's acting on $\mathbb{C}[\mathfrak{S}_n]$ and the elements $V \in \mathrm{Gr}_{n,2n}(\mathbb{C})$ with $\mathrm{Wr}(V) = (u + z_1) \cdots (u + z_n)$, sending the eigenvalue of β^λ to the Plücker coordinate $\Delta_{I_\lambda}(V)$.
(iii) If $z_1, \ldots, z_n \geq 0$, then the β^λ's are positive semidefinite.

Definition of the universal Plücker coordinates

$$\beta^\lambda := \sum_{\substack{X\subseteq\{1,\dots,n\},\\ |X|=|\lambda|}} \sum_{\pi\in\mathfrak{S}_X} \chi^\lambda(\pi)\pi \prod_{i\in[n]\setminus X} z_i \in \mathbb{C}[\mathfrak{S}_n]$$

- e.g. $n=2$. Write $\mathfrak{S}_2 = \{e,\sigma\}$, where e is the identity. We have

$$\beta^\varnothing = z_1z_2e, \quad \beta^{\square} = (z_1+z_2)e, \quad \beta^{\square\square} = e+\sigma, \quad \beta^{\substack{\square\\\square}} = e-\sigma,$$

and $\beta^\lambda = 0$ if $|\lambda| > 2$. On the eigenspace $\langle e-\sigma\rangle$, the eigenvalues are

$$\beta^\varnothing \rightsquigarrow z_1z_2, \quad \beta^{\square} \rightsquigarrow z_1+z_2, \quad \beta^{\square\square} \rightsquigarrow 0, \quad \beta^{\substack{\square\\\square}} \rightsquigarrow 2,$$

which are the Plücker coordinates of

$$V = \begin{bmatrix} \frac{z_1+z_2}{2} & 1 & 0 & 0 \\ -z_1z_2 & 0 & 2 & 0 \end{bmatrix} = \left\langle \frac{z_1+z_2}{2} + u,\ -z_1z_2 + u^2 \right\rangle \in \mathrm{Gr}_{2,4}(\mathbb{C}).$$

We can check that

$$\mathrm{Wr}(V) = \det \begin{bmatrix} \frac{z_1+z_2}{2} + u & -z_1z_2 + u^2 \\ 1 & 2u \end{bmatrix} = (u+z_1)(u+z_2).$$

Future directions

- Give combinatorial proofs of the commutativity relations and Plücker relations for the β^λ's.
- Find necessary and sufficient inequalities on the Plücker coordinates of V for all complex zeros of $\mathrm{Wr}(V)$ to be nonpositive. (The positivity conjecture implies that the inequalities $\Delta_I(V) \geq 0$ are necessary.)
- Address generalizations and variations of the Shapiro–Shapiro conjecture: the discriminant conjecture, the general form of the secant conjecture, the monotone conjecture, the total reality conjecture for convex curves, ...

Thank you!