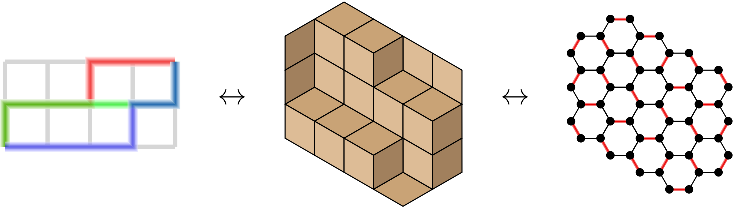
Combinatorics of the amplituhedron

Slides available at `www-personal.umich.edu/~snkarp`



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`arXiv:1608.08288` (joint with Lauren Williams)
`arXiv:1708.09525` (joint with Lauren Williams and Yan Zhang)

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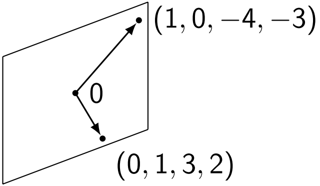
University of Massachusetts Amherst

The Grassmannian $\mathrm{Gr}_{k,n}$

- The *Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

$V :=$ 

$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$

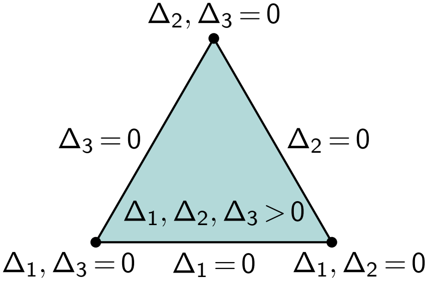
$$\Delta_{12} = 1, \;\; \Delta_{13} = 3, \;\; \Delta_{14} = 2, \;\; \Delta_{23} = 4, \;\; \Delta_{24} = 3, \;\; \Delta_{34} = 1$$

- Given $V \in \mathrm{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for k-subsets I of $\{1, \cdots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of V in columns I. The *Plücker coordinates* $\Delta_I(V)$ are well defined up to a common nonzero scalar.
- We call $V \in \mathrm{Gr}_{k,n}$ *totally nonnegative* if $\Delta_I(V) \geq 0$ for all k-subsets I. The set of all such V forms the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$.
- When $k = 1$, the Grassmannian $\mathrm{Gr}_{1,n}$ specializes to projective space $\mathbb{P}^{n-1}$, the set of nonzero vectors $(x_1 : \cdots : x_n)$ modulo rescaling.

The 'faces' of $\mathrm{Gr}_{k,n}^{\geq 0}$

- $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition due to Rietsch (1999) and Postnikov (2007). Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

$$\mathrm{Gr}_{1,3}^{\geq 0} = \mathbb{P}_{\geq 0}^{2} \cong$$



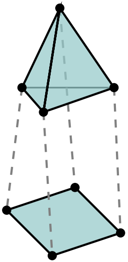
- $\mathrm{Gr}_{1,n}^{\geq 0}$ is an $(n-1)$-dimensional simplex in $\mathbb{P}^{n-1}$, since we can identify it with the set of points $(x_1 : \cdots : x_n)$ satisfying

$$x_1, \cdots, x_n \geq 0, \quad x_1 + \cdots + x_n = 1.$$

We can view $\mathrm{Gr}_{k,n}^{\geq 0}$ as a generalization of a simplex into the Grassmannian.

Amplituhedra and Grassmann polytopes

- By definition, a polytope is the image of a simplex under an affine map:



simplex in $\mathbb{P}^{n-1}$	$\dashrightarrow$	$\mathrm{Gr}_{k,n}^{\geq 0}$
$\downarrow$ linear map $\mathbb{R}^n \to \mathbb{R}^{m+1}$		$\downarrow$ linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$
polytope in $\mathbb{P}^m$	$\dashrightarrow$	*Grassmann polytope* in $\mathrm{Gr}_{k,k+m}$

A *Grassmann polytope* is the image of a map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

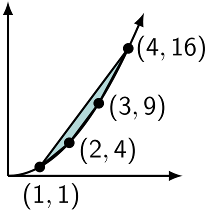
- When the matrix Z has positive maximal minors, the Grassmann polytope is called an *amplituhedron*, denoted $\mathcal{A}_{n,k,m}(Z)$. Amplituhedra generalize cyclic polytopes ($k = 1$) and totally nonnegative Grassmannians ($k + m = n$). They were introduced by Arkani-Hamed and Trnka (2014), and inspired Lam (2015) to define Grassmann polytopes.

Cyclic polytopes

- A *cyclic polytope* is a polytope (up to combinatorial equivalence) whose vertices line on the *moment curve*

$$f(t) := (t, t^2, \ldots, t^m) \text{ in } \mathbb{R}^m \quad (t > 0).$$

- e.g. $m = 2$



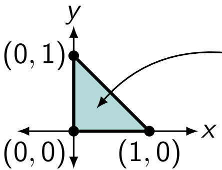
We can identify this polytope with the amplituhedron $\mathcal{A}_{4,1,2}(Z)$, where

$$Z = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix}.$$

- In general, any $k = 1$ amplituhedron $\mathcal{A}_{n,1,m}(Z)$ is a cyclic polytope. The columns of the $(m+1) \times n$ matrix Z give the vertices in $\mathbb{P}^m$.

Positive geometries and canonical forms

- Arkani-Hamed, Bai, Lam (2017): a *positive geometry* is a space equipped with a canonical differential form, which has logarithmic singularities at the boundaries of the space. Examples include convex polytopes:



$$\frac{dxdy}{xy(1-x-y)}$$

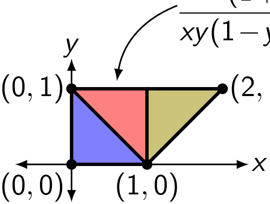
- $\mathrm{Gr}_{k,n}^{\geq 0}$ is a positive geometry. The canonical form of e.g. $\mathrm{Gr}_{2,4}^{\geq 0}$ is

$$\frac{dxdydzdw}{\Delta_{12}\Delta_{23}\Delta_{34}\Delta_{14}}, \text{ where } V = \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{bmatrix} \in \mathrm{Gr}_{2,4}\,.$$

- The amplituhedron is conjecturally a positive geometry, whose canonical form for $m = 4$ is the tree-level scattering amplitude in planar $\mathcal{N} = 4$ SYM.
- Other physically relevant positive geometries include *associahedra*, *cosmological polytopes*, *Cayley polytopes*, *halohedra*, *Stokes polytopes*, ...

Triangulations

- One way to find the canonical form of a positive geometry is by triangulating it into simpler pieces:



$$\frac{(1+y)dxdy}{xy(1-y)(1-x+y)} = \frac{dxdy}{xy(1-x-y)} + \frac{dxdy}{(1-x)(1-y)(x+y-1)} + \frac{dxdy}{(x-1)(1-y)(1-x+y)}$$

Conjecture (Arkani-Hamed, Trnka (2014))

The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is 'triangulated' by the images of certain $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, coming from the BCFW recursion.

Problem

Find a 'triangulation-independent' description of the amplituhedron form.

Sign variation

- As a simpler case, we studied the $m = 1$ amplituhedron $\mathcal{A}_{n,k,m}(Z)$. We also reformulated the definition of $\mathcal{A}_{n,k,m}(Z)$, in order to use *sign variation*.
- For $v \in \mathbb{R}^n$, let $\mathrm{var}(v)$ be the number of sign changes in the sequence $(v_1, v_2, \cdots, v_n)$, ignoring any zeros. Let $\overline{\mathrm{var}}(x)$ be the maximum number of sign changes we can get if we choose a sign for each zero component of x.

$$\mathrm{var}(1, -4, 0, -3, 6, 0, -1) = \mathrm{var}(1, -4, -3, 6, -1) = 3$$

$$\overline{\mathrm{var}}(1, -4, 0, -3, 6, 0, -1) = 5$$

Theorem (Gantmakher, Krein (1950))

Let $V \in \mathrm{Gr}_{k,n}$. The following are equivalent:
(i) V is totally nonnegative;
(ii) $\mathrm{var}(v) \le k - 1$ for all $v \in V \setminus \{0\}$;
(iii) $\overline{\mathrm{var}}(w) \ge k$ for all $w \in V^{\perp} \setminus \{0\}$.

- e.g. $\begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\ge 0}$.

Sign variation and the amplituhedron

- Recall: $\mathcal{A}_{n,k,m}(Z)$ is the image of $\mathrm{Gr}_{k,n}^{\geq 0}$ under the $(k+m)\times n$ matrix Z.
- Idea: rather than applying Z to the points $V \in \mathrm{Gr}_{k,n}^{\geq 0}$, we can dually intersect the orthogonal complement $V^{\perp}$ with the subspace $W \in \mathrm{Gr}_{k+m,n}$ spanned by the rows of Z. Since V is totally nonnegative and W is totally positive, we can apply the sign variation results of Gantmakher and Krein.
- In the case $m = 1$, we obtain the following:

Lemma (Karp, Williams)

$$\mathcal{A}_{n,k,1}(Z) \cong \{w \in \mathbb{P}(W) : \overline{\mathrm{var}}(w) = k\} \subseteq \mathbb{P}(W).$$

- In general, we can identify $\mathcal{A}_{n,k,m}(Z)$ with a subset of

$$\{X \in \mathrm{Gr}_m(W) : k \leq \overline{\mathrm{var}}(w) \leq k+m-1 \text{ for } w \in X \setminus \{0\}\}.$$

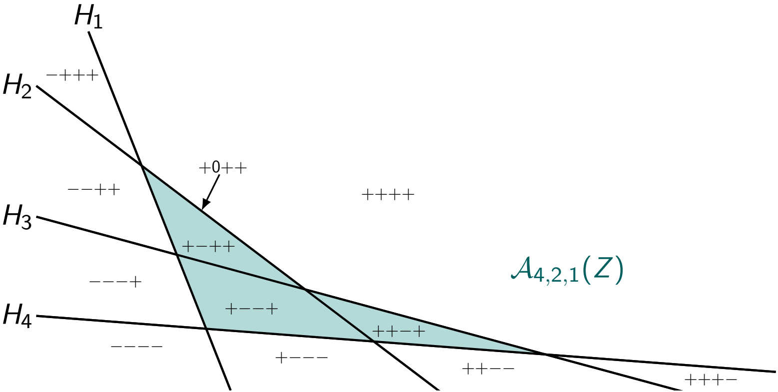
It is an open problem to determine whether this subset is in fact the whole set above. If so, this would give an 'intrinsic' definition of $\mathcal{A}_{n,k,m}(Z)$.
- Similar ideas were pursued by Arkani-Hamed, Thomas, and Trnka (2018).

Cyclic hyperplane arrangements

- A *cyclic hyperplane arrangement* consists of n hyperplanes of the form

$$tx_1 + t^2x_2 + \cdots + t^kx_k + 1 = 0 \text{ in } \mathbb{R}^k \quad (t > 0).$$

- e.g.



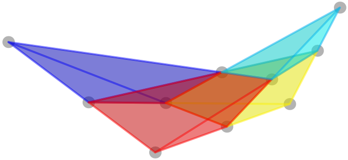
- Recall: $\mathcal{A}_{n,k,1}(Z) \cong \{w \in \mathbb{P}(W) : \overline{\text{var}}(w) = k\} \subseteq \mathbb{P}(W)$.

The $m = 1$ amplituhedron

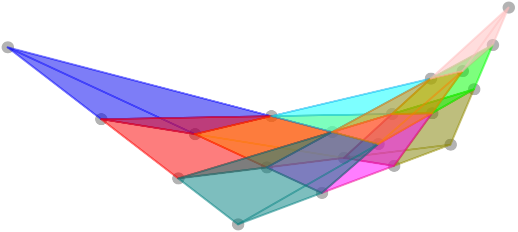
Theorem (Karp, Williams)

(i) $\mathcal{A}_{n,k,1}(Z)$ *is isomorphic to the complex of bounded faces of a cyclic hyperplane arrangement of n hyperplanes in* $\mathbb{R}^k$.
(ii) $\mathcal{A}_{n,k,1}(Z)$ *is isomorphic to a subcomplex of cells of* $\mathrm{Gr}_{k,n}^{\geq 0}$.
(iii) $\mathcal{A}_{n,k,1}(Z)$ *is homeomorphic to a closed ball of dimension k.*

- e.g.



$\mathcal{A}_{5,3,1}(Z)$



$\mathcal{A}_{6,3,1}(Z)$

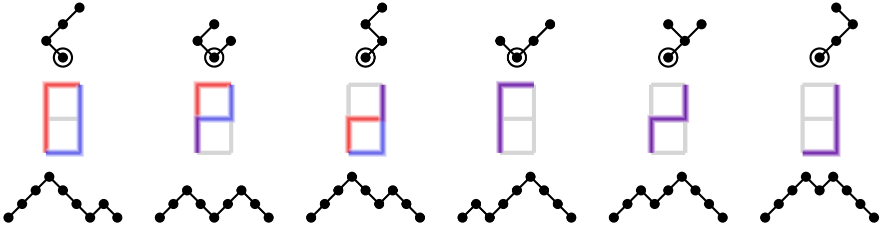
- Part (iii) follows from a general result of Dong about the bounded complex of a hyperplane arrangement in general position.

BCFW triangulation for $m = 4$

Conjecture (Arkani-Hamed, Trnka (2014))

The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is 'triangulated' by the images of certain $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, coming from the BCFW recursion.

- The number of top-dimensional cells in a BCFW triangulation is the *Narayana number* $N_{n-3,k+1} := \frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$.
- e.g. For $n = 7$, $k = 2$, we have $N_{7-3,2+1} = 6$:



- $k = 1$, m even: every triangulation of $\mathcal{A}_{n,1,m}(Z)$ has $\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$ top cells.
- $m = 2$: there is a nice triangulation of $\mathcal{A}_{n,k,2}(Z)$ with $\binom{n-2}{k}$ top cells.

Number of cells in a triangulation

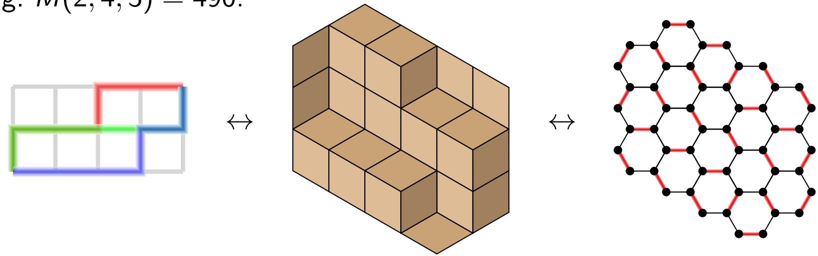
- Define the *MacMahon number*

$$M(a,b,c) := \prod_{p=1}^{a} \prod_{q=1}^{b} \prod_{r=1}^{c} \frac{p+q+r-1}{p+q+r-2}.$$

Conjecture (Karp, Williams, Zhang)

For m even, there exists a cell decomposition of $\mathcal{A}_{n,k,m}(Z)$ with $M(k, n-k-m, \frac{m}{2})$ top-dimensional cells.

- $M(a,b,c)$ is the number of *plane partitions* inside an $a \times b \times c$ box.
- e.g. $M(2,4,3) = 490$:



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Problem

Interpret properties of plane partitions in terms of amplituhedra.

- The $k \leftrightarrow n-k-m$ symmetry comes (for $m=4$) from *parity* of the scattering amplitude. Galashin and Lam (2018) showed that the *stacked twist map* interchanges triangulations of $\mathcal{A}_{n,k,m}(Z)$ and $\mathcal{A}_{n,n-k-m,m}(Z')$.

Problem

Explain the conjectural symmetry for amplituhedra between k and $\frac{m}{2}$.