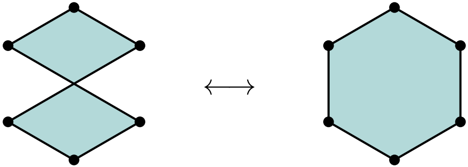
Gradient flows on totally nonnegative flag varieties

Slides available at `snkarp.github.io`



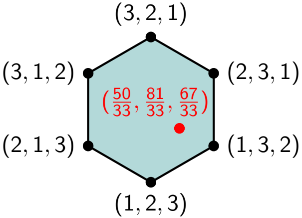
Steven N. Karp (University of Notre Dame)
joint work with Anthony M. Bloch

arXiv:2109.04558

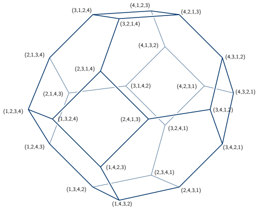
April 17th, 2023
Washington University

Schur–Horn theorem

- Let $\mathrm{Perm}(\lambda_1, \dots, \lambda_n)$ be the polytope in $\mathbb{R}^n$ whose vertices are all permutations of $(\lambda_1, \dots, \lambda_n)$, where $\lambda_1 > \dots > \lambda_n$.



$\mathrm{Perm}(3, 2, 1)$



$\mathrm{Perm}(4, 3, 2, 1)$

- Let μ send a matrix to its diagonal, e.g., $\mu\left(\frac{1}{33}\begin{bmatrix} 50 & 28 & 0 \\ 28 & 81 & 8 \\ 0 & 8 & 67 \end{bmatrix}\right) = \left(\frac{50}{33}, \frac{81}{33}, \frac{67}{33}\right)$.

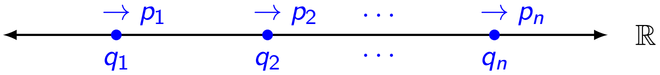
Theorem (Schur (1923), Horn (1953))

The map μ sends the space of $n \times n$ symmetric matrices with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ onto $\mathrm{Perm}(\lambda_1, \dots, \lambda_n)$.

Toda lattice

- The *Toda lattice* (1967) is a Hamiltonian system with

$$H(\mathbf{q},\mathbf{p}) := \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} \qquad \left(\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i}\right).$$



- Flaschka (1974) expressed the Toda flow in *Lax form*: $\dot{L} = [L, \pi_{\text{skew}}(L)]$, where L is an $n \times n$ symmetric tridiagonal matrix with positive subdiagonal.

$$L = \begin{bmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \\ 0 & a_2 & b_3 \end{bmatrix}, \quad \pi_{\text{skew}}(L) = \begin{bmatrix} 0 & -a_1 & 0 \\ a_1 & 0 & -a_2 \\ 0 & a_2 & 0 \end{bmatrix}, \quad a_i = \tfrac{1}{2}e^{\frac{q_i - q_{i+1}}{2}}, \quad b_i = -\tfrac{1}{2}p_i.$$

- The eigenvalues of L are distinct and invariant under the Toda flow. As $t \to \pm\infty$, L approaches a diagonal matrix with sorted diagonal entries.
- Let $\mathcal{J}_{\boldsymbol{\lambda}}^{>0}$ (respectively, $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$) denote the manifold of all L with fixed spectrum $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ and all $a_i > 0$ (respectively, $a_i \geq 0$).

Isospectral manifold $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$

Theorem (Moser (1975))

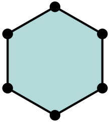
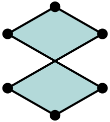
The map which sends $L \in \mathcal{J}_{\boldsymbol{\lambda}}^{>0}$ to the vector of first entries of its normalized eigenvectors is a homeomorphism onto $S_{>0}^{n-1}$.

- e.g. $L = \frac{1}{33}\begin{bmatrix} 50 & 28 & 0 \\ 28 & 81 & 8 \\ 0 & 8 & 67 \end{bmatrix} = \begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix}$

$$\mapsto \left(\tfrac{16}{33}, \tfrac{7}{33}, \tfrac{28}{33}\right) \in S_{>0}^{2}.$$

Theorem (Tomei (1984))

The space $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$ is homeomorphic to $\mathrm{Perm}(\boldsymbol{\lambda})$.

- However, $\mu : \mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0} \to \mathrm{Perm}(\boldsymbol{\lambda})$ is neither injective nor surjective.
- e.g. $\mathrm{Perm}(3,2,1) =$  , $\mu(\mathcal{J}_{(3,2,1)}^{\geq 0}) =$  .

Isospectral manifold of Jacobi matrices

Theorem (Bloch, Flaschka, Ratiu (1990))

Let Λ denote the diagonal matrix with diagonal $\boldsymbol{\lambda}$. Then the map

$$L = g\Lambda g^{-1} \mapsto \mu(g^{-1}\Lambda g) \qquad (g \in \mathrm{O}_n)$$

is a homeomorphism $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0} \to \mathrm{Perm}(\boldsymbol{\lambda})$, and is a diffeomorphism on $\mathcal{J}_{\boldsymbol{\lambda}}^{>0}$.

- e.g. $L = \begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix}$

$$\mapsto \mu\left(\begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \right) = \left(\tfrac{795}{363}, \tfrac{401}{363}, \tfrac{982}{363}\right).$$

- A key to the proof is to define a map $L = g\Lambda g^{-1} \mapsto g^{-1}\Lambda g$ on $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$, by choosing $g \in \mathrm{O}_n$ depending smoothly on L. We show that total positivity provides a natural way to choose g.

Totally nonnegative flag variety

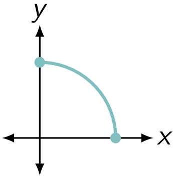
- The *complete flag variety* $\mathrm{Fl}_n(\mathbb{C})$ consists of all tuples of subspaces $V = (V_1, \dots, V_{n-1})$ of $\mathbb{C}^n$, where

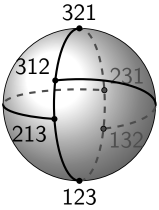
$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \quad \text{and} \;\; \dim(V_k) = k \text{ for all } k.$$

- We say that $g \in \mathrm{GL}_n(\mathbb{C})$ *represents* $V \in \mathrm{Fl}_n(\mathbb{C})$ if each V_k is the span of the first k columns of g. We call V *totally positive* if it is represented by some g whose left-justified (i.e. initial) minors are all real and positive. We denote the set of such V by $\mathrm{Fl}_n^{>0}$. We similarly define $\mathrm{Fl}_n^{\geq 0}$.

- e.g. $\begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{4} & 1 & 0 \\ \frac{7}{16} & \frac{17}{4} & 1 \end{bmatrix} \in \mathrm{Fl}_3^{>0}.$

$\mathrm{Fl}_2^{\geq 0}$





$\mathrm{Fl}_3^{\geq 0}$

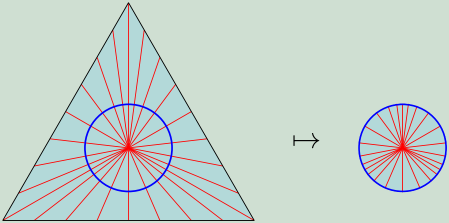
Topology of totally nonnegative flag varieties

Theorem (Galashin, Karp, Lam (2019))

The space $\mathrm{Fl}_n^{\geq 0}$ *is homeomorphic to a closed ball.*

Proof

Let M be the $n \times n$ tridiagonal matrix $\begin{bmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$. Then $V \mapsto \exp(tM)V$ for $t \in [0, \infty]$ contracts $\mathrm{Fl}_n^{\geq 0}$ onto a unique attractor in the interior.



Totally nonnegative adjoint orbit

- Let U_n be the group of $n \times n$ unitary matrices and $\mathfrak{u}_n$ its Lie algebra of $n \times n$ skew-Hermitian matrices. For $\lambda_1 > \cdots > \lambda_n$, consider the adjoint orbit

$$\mathcal{O}_{\boldsymbol{\lambda}} := \{g(\mathrm{i}\Lambda)g^{-1} : g \in \mathrm{U}_n\} \subseteq \mathfrak{u}_n, \quad \text{where } \Lambda := \mathrm{Diag}(\lambda_1, \dots, \lambda_n).$$

We have the isomorphism

$$\mathcal{O}_{\boldsymbol{\lambda}} \to \mathrm{Fl}_n(\mathbb{C}), \quad g(\mathrm{i}\Lambda)g^{-1} \mapsto g,$$

sending a matrix to its flag of eigenvectors ordered by descending eigenvalue.

- We define $\mathcal{O}_{\boldsymbol{\lambda}}^{>0}$ and $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ to be the preimages of $\mathrm{Fl}_n^{>0}$ and $\mathrm{Fl}_n^{\geq 0}$.
- e.g. $\begin{bmatrix} \frac{16}{33} & \frac{7}{33} & \frac{28}{33} \\ \frac{28}{33} & \frac{4}{33} & -\frac{17}{33} \\ \frac{7}{33} & -\frac{32}{33} & \frac{4}{33} \end{bmatrix} \begin{bmatrix} 3\mathrm{i} & 0 & 0 \\ 0 & 2\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{bmatrix} \begin{bmatrix} \frac{16}{33} & \frac{28}{33} & \frac{7}{33} \\ \frac{7}{33} & \frac{4}{33} & -\frac{32}{33} \\ \frac{28}{33} & -\frac{17}{33} & \frac{4}{33} \end{bmatrix} = \frac{\mathrm{i}}{33} \begin{bmatrix} 50 & 28 & 0 \\ 28 & 81 & 8 \\ 0 & 8 & 67 \end{bmatrix} \in \mathcal{O}_{(3,2,1)}^{>0}.$

Proposition (Bloch, Karp (2023))

The tridiagonal subset of $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ is precisely $\mathrm{i}\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$ (i.e. where all off-diagonal entries lie on the nonnegative imaginary axis).

Gradient flows on adjoint orbits

- We consider the gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ of the function $L \mapsto 2n\,\mathrm{tr}(LN)$, where $N \in \mathfrak{u}_n$. We work in the *Kähler*, *normal*, and *induced* metrics.
- We say that the flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ *strictly preserves positivity* if trajectories starting in $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ lie in $\mathcal{O}_{\boldsymbol{\lambda}}^{>0}$ for all positive time. If so, we obtain a contractive flow with the Lyapunov function $L \mapsto -2n\,\mathrm{tr}(LN)$.

Proposition (Duistermaat, Kolk, Varadarajan (1983); Guest, Ohnita (1993))

The isomorphism $\mathcal{O}_{\boldsymbol{\lambda}} \cong \mathrm{Fl}_n(\mathbb{C})$ sends the gradient flow with respect to N in the Kähler metric to the flow $V(t) = \exp(t\mathrm{i}N)V$ on $\mathrm{Fl}_n(\mathbb{C})$.

Theorem (Bloch, Karp (2023))

The gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ with respect to N in the Kähler metric strictly preserves positivity if and only if $\mathrm{i}N \in \mathcal{J}_{\boldsymbol{\mu}}^{>0}$ for some $\boldsymbol{\mu}$.

- The contractive flow on $\mathrm{Fl}_n^{\geq 0}$ considered earlier is such a flow. We also obtain contractive flows on a new family of *amplituhedra*.

Gradient flows: normal and induced metrics

Proposition (Brockett (1991); Bloch, Brockett, Ratiu (1992))

The gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ with respect to N in the normal metric is

$$\dot{L} = [L, [L, N]].$$

Theorem (Bloch, Karp (2023))

No gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ in the normal metric strictly preserves positivity.

Proposition (Bloch, Karp (2023))

The gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ with respect to N in the induced metric is

$$\dot{L} = [L, \mathrm{ad}_L^{-1}(N)].$$

Proposition (Bloch, Karp (2023))

Let $\lambda_1 > \lambda_2 > \lambda_3$ satisfy $\frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_3} \notin [\frac{1}{2+2\sqrt{2}}, 2 + 2\sqrt{2}]$. Then no gradient flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ in the induced metric strictly preserves positivity.

Twist map

- Every element of $\mathrm{Fl}_n^{\geq 0}$ is represented by a unique $g \in \mathrm{U}_n$ whose left-justified minors are all nonnegative. Let $\vartheta(g) := ((-1)^{i+j}(g^{-1})_{i,j})_{1 \leq i,j \leq n}$.
- e.g. $\vartheta\left(\frac{1}{33}\begin{bmatrix} 16 & -7 & 28 \\ 28 & -4 & -17 \\ 7 & 32 & 4 \end{bmatrix}\right) = \frac{1}{33}\begin{bmatrix} 16 & -28 & 7 \\ 7 & -4 & -32 \\ 28 & 17 & 4 \end{bmatrix} \overset{\mathrm{Fl}_n}{\equiv} \begin{bmatrix} 16 & 16 \cdot 3 & 16 \cdot 3^2 \\ 7 & 7 \cdot 2 & 7 \cdot 2^2 \\ 28 & 28 \cdot 1 & 28 \cdot 1^2 \end{bmatrix}$.

Theorem (Bloch, Karp (2023))

The involution ϑ defines a diffeomorphism $\mathrm{Fl}_n^{\geq 0} \to \mathrm{Fl}_n^{\geq 0}$.

- The map ϑ induces a map on $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$. Restricting to $\mathrm{i}\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$, we recover the map of Bloch, Flaschka, and Ratiu on $\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$:

$$L = g\Lambda g^{-1} \mapsto g^{-1}\Lambda g, \qquad \text{where } \Lambda := \mathrm{Diag}(\lambda_1, \dots, \lambda_n).$$

Proposition (Bloch, Karp (2023))

For $x \in \mathbb{R}^n_{>0}$, let $\mathrm{Vand}(\boldsymbol{\lambda}, x) \in \mathrm{Fl}_n(\mathbb{C})$ be the complete flag generated by $x, \Lambda x, \dots, \Lambda^{n-1}x$. Then the image of $\mathrm{i}\mathcal{J}_{\boldsymbol{\lambda}}^{>0} \subseteq \mathcal{O}_{\boldsymbol{\lambda}}^{>0} \cong \mathrm{Fl}_n^{>0}$ is

$$\vartheta(\{\mathrm{Vand}(\boldsymbol{\lambda}, x) : x \in \mathbb{R}^n_{>0}\}) \subseteq \mathrm{Fl}_n^{>0}.$$

Context for the twist map ϑ

1. The twist map ϑ is motivated by the twist map on $\mathrm{GL}_n(\mathbb{C})$ of Berenstein, Fomin, and Zelevinsky, but based on the Iwasawa (or QR-) decomposition of $\mathrm{GL}_n(\mathbb{C})$, rather than the Bruhat decomposition.

- e.g. $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ b & a & 1 \end{bmatrix}$ (BFZ-style twist on $\mathrm{Fl}_3(\mathbb{C})$)

2. We have $\mathrm{Fl}_n(\mathbb{C}) = K/T$, where $K := \mathrm{U}_n(\mathbb{C})$ and $T \subseteq \mathrm{U}_n(\mathbb{C})$ is the subset of diagonal matrices. We have the isomorphism

$$K/T \xrightarrow{\cong} T\backslash K, \quad gT \mapsto (gT)^{-1} = Tg^{-1}.$$

What if we want an isomorphism of the form

$$K/T \xrightarrow{\cong} T\backslash K, \quad gT \mapsto Tg?$$

(This arose in work of Ayzenberg and Buchstaber on Hessenberg varieties.) This depends on the choice of $g \in K$ for the given T-coset, and ϑ gives a smooth choice of g on $\mathrm{Fl}_n^{\geq 0}$ (unlike BFZ-style twists).

Toda flow and total positivity

- Recall the Toda flow on symmetric tridiagonal matrices:

$$\dot{L} = [L, \pi_{\mathrm{skew}}(L)], \quad L \in \mathcal{J}_{\boldsymbol{\lambda}}^{>0}.$$

Repacling L by $-\mathrm{i}L \in \mathcal{O}_{\boldsymbol{\lambda}}$, we obtain the *full symmetric Toda flow* on $\mathcal{O}_{\boldsymbol{\lambda}}$:

$$\dot{L} = [L, \pi_{\mathfrak{u}_n}(-\mathrm{i}L)], \quad L \in \mathcal{O}_{\boldsymbol{\lambda}}.$$

- Bloch (1990): The tridiagonal Toda flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ is the gradient flow with respect to $N = -\mathrm{i}\,\mathrm{Diag}(n-1, \dots, 1, 0)$ in the normal metric.

Theorem (Bloch, Karp (2023))

The full symmetric Toda flow on $\mathcal{O}_{\boldsymbol{\lambda}}$ weakly preserves positivity, and is the twisted gradient flow with respect to $N = -\mathrm{i}\Lambda$ in the Kähler metric.

- Gekhtman and Shapiro (1997) and Kodama and Williams (2015) proved related results for the *full Kostant–Toda flow* on Hessenberg matrices.

- e.g. $n = 3$: $\dot{L} = [L, L_-]$, where $L = \begin{bmatrix} a_{1,1} & 1 & 0 \\ a_{2,1} & a_{2,2} & 1 \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$, $L_- = \begin{bmatrix} 0 & 0 & 0 \\ a_{2,1} & 0 & 0 \\ a_{3,1} & a_{3,2} & 0 \end{bmatrix}$.

Future directions

- Study flows on the cell closures of the cell decomposition of $\mathrm{Fl}_n^{\geq 0}$.
- Study the subset of $\mathcal{O}_{\boldsymbol{\lambda}}^{\geq 0}$ with bounded bandwidth (the tridiagonal subset being $\mathrm{i}\mathcal{J}_{\boldsymbol{\lambda}}^{\geq 0}$).
- Extend the setup beyond type A.
- Generalize the connection to Toda flows to the periodic and continuous cases.
- Study Toda flows on permutohedra.

Thank you!