q-Whittaker functions, finite fields, and Jordan forms

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A partition $\lambda$ is a weakly-decreasing sequence of nonnegative integers.

e.g. $\lambda = (4, 4, 1) = \begin{array}{cccc}
\text{3} & \text{3} & \text{4} & \\
\text{4} & \text{4} & \text{8} & \\
\text{5} & 
\end{array}$

A semistandard tableau $T$ is a filling of $\lambda$ with positive integers which is weakly increasing across rows and strictly increasing down columns.

Definition (Schur function)

$$s_\lambda(x_1, x_2, \ldots) := \sum_T x^T,$$

where the sum is over all semistandard tableaux $T$ of shape $\lambda$.

$s_\lambda(x)$ is symmetric in the variables $x_i$. 
Schur functions

- e.g. \( s_{(2,1)}(x_1, x_2, x_3) = \)
  \[
  x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2
  \]

Schur functions appear in many contexts; for example, they:

- form an *orthonormal basis* for the algebra of symmetric functions in \( x \);
- are characters of the *irreducible polynomial representations* of \( GL_n(\mathbb{C}) \);
- give the values of the *irreducible characters* of the symmetric group \( S_n \), when expanded in terms of power sum symmetric functions;
- are representatives for *Schubert classes* in the cohomology ring of the Grassmannian \( Gr_{k,n}(\mathbb{C}) \);
- define the *Schur processes* of Okounkov and Reshetikhin (2003).
Cauchy identity

Theorem (Cauchy)

\[
\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x)s_\lambda(y)
\]

We can expand the left-hand side as a sum indexed by nonnegative-integer matrices, and the right-hand side as a sum indexed by pairs of semistandard tableaux of the same shape.

e.g. Taking the coefficient of \(x_1 x_2 y_1 y_2\) on each side gives

\[
1 + 1 = 1 + 1
\]

\[
12 + 21 = \left(\begin{array}{c}1 \\ 2\end{array}, \begin{array}{c}1 \\ 2\end{array}\right) + \left(\begin{array}{c}12 \\ 12\end{array}\right)
\]
Burge correspondence (1974)

- The Burge correspondence (also known as column Robinson–Schensted–Knuth) is a bijection

\[ M \leftrightarrow (P(M), Q(M)) \]

between nonnegative-integer matrices and pairs of semistandard tableaux of the same shape. It proves the Cauchy identity for Schur functions.

- P(M) is obtained via column insertion and Q(M) is obtained via recording.

- e.g. \( w = 25143 \)

\[
\begin{array}{cccccccc}
2 & 2 & 1 & 1 & 1 & 2 & 5 & 3 \\
5 & 5 & 4 & 5 & 3 & 4 & 2 & 4 \\
\end{array}
\]

\[ P(w) \quad \quad \quad \quad Q(w) \]
Nilpotent matrices

- An $n \times n$ matrix $N$ over $\mathbb{k}$ is *nilpotent* if some power of $N$ is zero. Such an $N$ can be conjugated over $\mathbb{k}$ into *Jordan form*. Let $JF^\top(N)$ be the *transpose* of the partition given by the sizes of the Jordan blocks.

- e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- Algebraically, $JF^\top(N)$ is the partition $\lambda$ given by

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i = \text{dim}(\ker(N^i)) \quad \text{for all } i.$$ 

**Theorem (Gansner (1981))**

Let $N$ be a generic $n \times n$ strictly upper-triangular matrix, where $N_{i,j} = 0$ for all inversions $(i,j)$ of $w^{-1}$. Then $P(w)$ and $Q(w)$ can be read off from the Jordan forms of the leading submatrices of $N$ and $w^{-1}Nw$. 
Burge correspondence via Jordan forms

e.g. \( w = 25143 \)

\[ N = \begin{bmatrix}
0 & 0 & a & b & 0 \\
0 & 0 & c & d & e \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

\((a, b, c, d, e \in \mathbb{k} \text{ generic})\)

\[ P(w): \]

\[ Q(w): \]
Flag variety

- A complete flag $F$ in $\mathbb{k}^n$ is a sequence of nested subspaces
  
  $$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = \mathbb{k}^n, \quad \dim(F_i) = i \text{ for all } i.$$

- An $n \times n$ (nilpotent) matrix $N$ is strictly compatible with $F$ if
  
  $$N(F_i) \subseteq F_{i-1} \quad \text{for all } i.$$

- The $N$ in Gansner’s theorem is precisely a matrix strictly compatible with two complete flags $F$ and $F'$ defined by
  
  $$F_i := \langle e_1, e_2, \ldots, e_i \rangle \quad \text{and} \quad F'_j := \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(j)} \rangle.$$

The two sequences of matrices in the theorem are $(N|_{F_i})_{i=1}^n$ and $(N|_{F'_j})_{j=1}^n$.

- More generally, we can take any pair of flags $(F, F')$ with relative position $w$, denoted $F \overset{w}{\longrightarrow} F'$. The relative position records $\dim(F_i \cap F'_j)$ for all $i$ and $j$, or alternatively, the Schubert cell of $F'$ relative to $F$. 

**Theorem (Steinberg (1976, 1988), Spaltenstein (1982), Rosso (2012))**

Fix partial flags $F$ and $F'$ with $F \xrightarrow{M} F'$. Let $N$ be a generic nilpotent matrix strictly compatible with both $F$ and $F'$. Then

$$P(M) = JF^\top(N; F) \quad \text{and} \quad Q(M) = JF^\top(N; F').$$

- If $F \xrightarrow{w} F'$, then $F' \xrightarrow{w^{-1}} F$. This implies the symmetry

  $$P(w^{-1}) = Q(w).$$

- What happens when $k$ is a finite field, and we consider all choices of $N$ (not necessarily generic)?
**q-Whittaker functions**

- Define \([n]_q := 1 + q + q^2 + \cdots + q^{n-1}\) and \([n]_q! := [n]_q[n-1]_q \cdots [1]_q\).

**Definition (q-Whittaker function)**

\[
W_\lambda(x_1, x_2, \ldots ; q) := \sum_T \text{wt}_q(T)x^T,
\]

where the sum is over all semistandard tableaux \(T\) of shape \(\lambda\).

- \(W_\lambda(x; q)\) is symmetric in the variables \(x_i\), and specializes to \(s_\lambda(x)\) when \(q = 0\). We obtain the \(\mathfrak{gl}_n\)-Whittaker functions as a certain \(q \to 1\) limit.

- e.g. \(T = \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 7 \\
6 & & 
\end{array}\) \quad \text{wt}_q(T) = [1]_q[2]_q[1]_q[2]_q[2]_q[1]_q[2]_q = (1 + q)^4

- We have the following specializations:

\[
W_\lambda(x; q) = P_\lambda(x; q, 0) = q^{\deg(\tilde{H}_\lambda)}\omega(\tilde{H}_\lambda(x; 1/q, 0)), \quad W_\lambda(x; 1) = e_{\lambda^\top}(x).
\]
Theorem (Macdonald (1995))

\[
\prod_{i,j \geq 1} \prod_{d \geq 0} \frac{1}{1 - x_i y_j q^d} = \sum_{\lambda} \frac{(1 - q)^{-\lambda_1}}{\prod_{i \geq 1} [\lambda_i - \lambda_{i+1}]q!} W_\lambda(x; q) W_\lambda(y; q)
\]

- This gives the *partition function* for the *q-Whittaker processes*, a special case of the *Macdonald processes* of Borodin and Corwin (2014).
- e.g. Taking the coefficient of \(x_1 x_2 y_1 y_2\) on each side gives

\[
(1 - q)^{-2} + (1 - q)^{-2} = (1 - q)^{-1} + (1 - q)^{-2}(1 + q)
\]

\[
\begin{align*}
12 & 21 \\
\begin{pmatrix} \frac{1}{2} \end{pmatrix} & \begin{pmatrix} \frac{1}{2} \end{pmatrix} \\
\begin{pmatrix} 1 \ 2 \end{pmatrix} & \begin{pmatrix} 1 \ 2 \end{pmatrix}
\end{align*}
\]
\[ N = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad (a \in \mathbb{F}_{1/q}) \]

- e.g. \( w = 12 \)
  \[ P(w): \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad Q(w): \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

- \( a \neq 0 \):
  \[ a = 0: \begin{bmatrix} \hline 1 \\ 2 \end{bmatrix} \]

\[ P = 1 - q \]

\[ P = q \]

- e.g. \( w = 21 \)
  \[ N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ P(w): \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad Q(w): \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

\[ P = 1 \]
Let $1/q$ be a prime power, and fix partial flags $F \xrightarrow{M} F'$ over $\mathbb{F}_{1/q}$. Let $N$ denote a uniformly random nilpotent matrix strictly compatible with both $F$ and $F'$. For semistandard tableaux $T$ and $T'$ of the same shape, define

$$p_M(T, T') := \mathbb{P}[JF^T(N; F) = T \text{ and } JF^T(N; F') = T'].$$

(This definition depends only on $M$, not on the choice of $(F, F')$.)

\textbf{Theorem (Karp, Thomas (2021+))}

(i) The maps $p_M(\cdot, \cdot)$ define a probabilistic bijection proving the Cauchy identity for $q$-Whittaker functions, called the $q$-Burge correspondence.

(ii) As $q \to 0$, the $q$-Burge correspondence converges to the deterministic Burge correspondence.

It is an open problem to determine if $p_M(T, T')$ is a polynomial in $q$.

A different probabilistic bijection was given by Matveev and Petrov (2017), using the $q$-row insertion of Borodin and Petrov (2016).
Proof outline

**Theorem (Karp, Thomas (2021+))**

Fix a nilpotent matrix $N$ over $\mathbb{F}_{1/q}$ with Jordan type $\lambda$. The coefficient of $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ in $W_\lambda(x; q)$ equals $q \sum_i \binom{\lambda_i}{2} - \binom{\alpha_i}{2}$ times the number of partial flags $F$ over $\mathbb{F}_{1/q}$ strictly compatible with $N$ satisfying

$$\dim(F_i) = \alpha_1 + \cdots + \alpha_i \quad \text{for all } i.$$ 

- e.g. $\lambda = [\square\square], \; N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then the coefficient of $x_1x_2$ in $W_\lambda(x; q)$ is

$$q^1 \cdot \#(\text{complete flags in } \mathbb{F}_{1/q}^2) = q(1 + 1/q) = q + 1.$$ 

- This is similar to a formula for the modified Hall–Littlewood functions $\tilde{H}_\lambda(x; q, 0)$ in terms of weakly compatible flags over $\mathbb{F}_q$.

- To prove that the $q$-Burge correspondence works, we double count

$$\{(F, F', N) : F \xrightarrow{M} F', \ JF^T(N; F) = T, \ JF^T(N; F') = T'\}.$$ 

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Consider a path quiver with a unique sink:

\[ Q = \]

\[ \begin{array}{c}
\bullet \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\bullet \\
\end{array} \]

A representation \( V \) of \( Q \) is an assignment of a vector space to each vertex and a linear map to each arrow, e.g.,

\[
V = \begin{array}{ccc}
\mathbb{k} & [-1] & \mathbb{k} \\
\rightarrow & [3] & \mathbb{k}^2 \\
\rightarrow & [2] & \mathbb{k}^2 \\
\rightarrow & -1 & 0 \\
\rightarrow & 0 & 0 \\
\end{array}
\]

We will only consider \( V \) where every linear map is injective. Isomorphism classes of such \( V \) are indexed by nonnegative-integer matrices \( M \).

We now decorate \( V \) with a linear map for the reverse of each arrow, such that a relation holds for every vertex:

\[ \alpha \circ \gamma \pm \delta \circ \beta = 0 \]

This defines a module \( V^\# \) over the preprojective algebra of \( Q \).
Socle filtration

- Up to isomorphism, $V^\#$ is given (non-uniquely) by a triple $(F, F', N)$:

$$V^\# = \begin{array}{ccccccc} F'_1 & \text{id} & F'_2 & \text{id} & F'_3 = F_3 & \text{id} & F_2 & \text{id} & F_1 \\ -N & & -N & & N & & N & \\ \end{array}$$

- The socle filtration of $V^\#$ corresponds precisely to the pair of tableaux $(T, T') = (\text{JF}^T(N; F), \text{JF}^T(N; F'))$.

- e.g.

$$V^\# = \begin{array}{ccc} \langle e_1 \rangle & \text{id} & \langle e_1 \rangle \\ \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \end{array}$$

$$\longleftrightarrow \left( \begin{bmatrix} 1 & \\ 2 & 1 \\ 2 & \\ \end{bmatrix}, \begin{bmatrix} 1 & \\ 2 & 1 \\ 2 & \\ \end{bmatrix} \right)$$
The $q$-Burge correspondence gives enumerative results such as:

**Theorem (Karp, Thomas (2021+))**

Let $(T, T')$ be a pair of semistandard tableaux of shape $\lambda$, and let $d$ be a dimension vector of $Q$. Then

$$
\sum_{V^\#} \frac{1}{|\text{Aut}(V^\#)|} = \frac{q^{c(d)}(1 - q)^{-\lambda_1}}{\prod_{i \geq 1} [\lambda_i - \lambda_{i+1}]_q!} \text{wt}_q(T) \text{wt}_q(T'),
$$

where the sum is over all $V^\#$ over $\mathbb{F}_{1/q}$ up to isomorphism, with dimension vector $d$ and socle filtration corresponding to $(T, T')$.

The description of $W_\lambda(x; q)$ in terms of partial flags can be rephrased as enumerative statements for certain Nakajima quiver varieties over $\mathbb{F}_{1/q}$.

Thank you!